KENDRIYA VIDYALAYA SANGATHAN

ZONAL INSTITUTE OF EDUCATION AND TRAINING MYSURU

STUDY MATERIAL

APPLIED MATHEMATICS

CLASS XII

2023 - 24

पाठ्यक्रम निदेशक / COURSE DIRECTOR

सुश्री मीनाक्षी जैन/Ms. MENAXI JAIN

के. वि. संगठन,शिक्षा एवं प्रशिक्षण का आंचलिक संस्थान, मैसूरू

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Co-ordinated by: Mr. D SREENIVASULU,
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DIRECTOR'S MESSAGE……

It is with profound delight and utmost pride that I announce the publication of our study material of CLASS **XII (APPLIED MATHEMATICS)** for the session 2023-24. It's my firm belief that access to quality education should know no boundaries, transcending social and economic constraints. Our collective vision is to empower all students with the tools for success and intellectual growth.

With their steadfast dedication PGT-MATHEMATICS of Bangalore, Chennai, Ernakulam & Hyderabad regions of Kendriya Vidyalaya Sangathan have invested their knowledge, expertise, and passion into meticulously crafting these study materials to complement the classroom learning experience of the students. These materials serve as invaluable aids for self-study since they are comprehensive, well-structured, and presented in a manner that is easy to comprehend.

It is with pleasure that I place on record my commendation for the commitment and dedication of the team of teachers which included Mr. D. SREENIVASULU, Training Associate (MATHEMATICS) from ZIET Mysore who has been the Coordinator of this assignment and all the concerned PGT- Mathematics subject experts from the four feeder regions of ZIET Mysore.

Wishing you all the very best in your academic journey!

MENAXI JAIN
DIRECTOR
ZIET MYSORE

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INDEX

CURRICULUM

APPLIED MATHEMATICS (CODE – 241)

CLASS XII

SESSION: 2023-24

Number of Paper: 1

Total number of Periods: 240 (35 Minutes Each)

Time:3 Hours

Max Marks:80

No.	Units	No. of Periods	Marks
I	Numbers, Quantification and Numerical Applications	30	11
II	Algebra	20	10
III	Calculus	50	15
IV	Probability Distributions	35	10
V	Inferential Statistics	10	05
VI	Index Numbers and Time-based data	30	06
VII	Financial Mathematics	50	15
VIII	Linear Programming	15	08
Total		240	80
Internal Assessment			20

CLASS XII

Sl. No.	Contents	Learning Outcomes: Students will be able to	Notes / Explanation
UNIT-1 NUMBERS, QUANTIFICATION AND NUMERICAL APPLICATIONS			
1.1	Modulo Arithmetic	• Define modulus of an integer • Apply arithmetic operations using modular arithmetic rules	• Definition and meaning • Introduction to modulo operator • Modular addition and subtraction
1.2	Congruence Modulo	• Define congruence modulo • Apply the definition in various problems	• Definition and meaning • Solution using congruence modulo • Equivalence class
1.4	Alligation and Mixture	• Understand the rule of alligation to produce a mixture at a given price • Determine the mean price of a mixture • Apply rule of alligation	• Meaning and Application of rule of alligation • Mean price of a mixture
1.5	Numerical Problems	Solve real life problems mathematically	
	Boats and Streams (upstream and downstream)	• Distinguish between upstream and downstream • Express the problem in the form of an equation	• Problems based on speed of stream and the speed of boat in still water
	Pipes and Cisterns	• Determine the time taken by two or more pipes to fill or empty the tank	• Calculation of the portion of the tank filled or drained by the pipe(s) in unit time
	Races and Games	• Compare the performance of two players w.r.t. time, distance	• Calculation of the time taken/ distance covered / speed of each player
1.6	Numerical Inequalities	• Describe the basic concepts of numerical inequalities • Understand and write numerical inequalities	• Comparison between two statements/situations which can be compared numerically • Application of the techniques of numerical solution of algebraic inequations
UNIT-2 ALGEBRA			
2.1	Matrices and types of matrices	• Define matrix • Identify different kinds of matrices • Find the size / order of matrices	• The entries, rows and columns of matrices • Present a set of data in a matrix form
2.2	Equality of matrices, Transpose of a matrix, Symmetric and Skew symmetric matrix	• Determine equality of two matrices • Write transpose of given matrix • Define symmetric and skew symmetric matrix	• Examples of transpose of matrix • A square matrix as a sum of symmetric and skew symmetric matrix • Observe that diagonal elements of skew symmetric matrices are always zero

2.3	Algebra of Matrices	• Perform operations like addition & subtraction on matrices of same order • Perform multiplication of two matrices of appropriate order • Perform multiplication of a scalar with matrix	• Addition and Subtraction of matrices • Multiplication of matrices (It can be shown to the students that Matrix multiplication is similar to multiplication of two polynomials) • Multiplication of a matrix with a real number
2.4	Determinants	• Find determinant of a square matrix • Use elementary properties of determinants	• Singular matrix, Non-singular matrix • $\|AB\|=\|A\|\|B\|$ • Simple problems to find determinant value
2.5	Inverse of a matrix	• Define the inverse of a square matrix • Apply properties of inverse of matrices	• Inverse of a matrix using: a) cofactors If A and B are invertible square matrices of same size, i) $(AB)^{-1}=B^{-1}A^{-1}$ ii) $(A^{-1})^{-1}=A$ iii) $(A^{T})^{-1}=(A^{-1})^{T}$
2.6	Solving system of simultaneous equations using matrix method, Cramer's rule and	• Solve the system of simultaneous equations using i) Cramer's Rule ii) Inverse of coefficient matrix • Formulate real life problems into a system of simultaneous linear equations and solve it using these methods	• Solution of system of simultaneous equations upto three variables only (non- homogeneous equations)
UNIT- 3 CALCULUS			
Differentiation and its Applications			
3.1	Higher Order Derivatives	• Determine second and higher order derivatives • Understand differentiation of parametric functions and implicit functions	• Simple problems based on higher order derivatives • Differentiation of parametric functions and implicit functions (upto 2nd order)
3.2	Application of Derivatives	• Determine the rate of change of various quantities • Understand the gradient of tangent and normal to a curve at a given point • Write the equation of tangents and normal to a curve at a given point	• To find the rate of change of quantities such as area and volume with respect to time or its dimension • Gradient / Slope of tangent and normal to the curve • The equation of the tangent and normal to the curve (simple problems only)
3.3	Marginal Cost and Marginal Revenue using derivatives	• Define marginal cost and marginal revenue • Find marginal cost and marginal revenue	• Examples related to marginal cost, marginal revenue, etc.

3.4	Increasing /Decreasing Functions	• Determine whether a function is increasing or decreasing • Determine the conditions for a function to be increasing or decreasing	• Simple problems related to increasing and decreasing behaviour of a function in the given interval
3.5	Maxima and Minima	• Determine critical points of the function • Find the point(s) of local maxima and local minima and corresponding local maximum and local minimum values • Find the absolute maximum and absolute minimum value of a function • Solve applied problems	• A point x= c is called the critical point of f if f is defined at c and $\mathrm{f}'(\mathrm{c}) = 0$ or f is not differentiable at c • To find local maxima and local minima by: i) First Derivative Test ii) Second Derivative Test • Contextualized real life problems
Integration and its Applications			
3.6	Integration	• Understand and determine indefinite integrals of simple functions as anti-derivative	• Integration as a reverse process of differentiation • Vocabulary and Notations related to Integration
3.7	Indefinite Integrals as family of curves	• Evaluate indefinite integrals of simple algebraic functions by method of: i) substitution ii) partial fraction iii) by parts	• Simple integrals based on each method (non-trigonometric function)
3.8	Definite Integrals as area under the curve	• Define definite integral as area under the curve • Understand fundamental theorem of Integral calculus and apply it to evaluate the definite integral • Apply properties of definite integrals to solve the problems	• Evaluation of definite integrals using properties
3.9	Application of Integration	• Identify the region representing C.S. and P.S. graphically • Apply the definite integral to find consumer surplus-producer surplus	Problems based on finding • Total cost when Marginal Cost is given • Total Revenue when Marginal Revenue is given • Equilibrium price and equilibrium quantity and hence consumer and producer surplus
Differential Equations and Modeling			
3.10	Differential Equations	• Recognize a differential equation • Find the order and degree of a differential equation	• Definition, order, degree and examples

3.11	Formulating and Solving Differential Equations	• Formulate differential equation • Verify the solution of differential equation • Solve simple differential equation	• Formation of differential equation by eliminating arbitrary constants • Solution of simple differential equations (direct integration only)
3.12	Application of Differential Equations	• Define Growth and Decay Model • Apply the differential equations to solve Growth and Decay Models	• Growth and Decay Model in Biological sciences, Economics and business, etc.
UNIT- 4 PROBABILITY DISTRIBUTIONS			
4.1	Probability Distribution	• Understand the concept of Random Variables and its Probability Distributions • Find probability distribution of discrete random variable	• Definition and example of discrete and continuous random variable and their distribution
4.2	Mathematical Expectation	• Apply arithmetic mean of frequency distribution to find the expected value of a random variable	• The expected value of discrete random variable as summation of product of discrete random variable by the probability of its occurrence.
4.3	Variance	• Calculate the Variance and S.D. of a random variable	• Questions based on variance and standard deviation
4.4	Binomial Distribution	• Identify the Bernoulli Trials and apply Binomial Distribution • Evaluate Mean, Variance and S.D of a binomial distribution	• Characteristics of the binomial distribution • Binomial formula: $P(r) = {}^nC_r\, p^r q^{n-r}$ Where n = number of trials P = probability of success q = probability of failure Mean $=np$ Variance = npq Standard Deviation = $\sqrt{npq}$
4.5	Poison Distribution	• Understand the Conditions of Poisson Distribution • Evaluate the Mean and Variance of Poisson distribution	• Characteristics of Poisson Probability distribution Poisson formula: $P(x) = \frac{\lambda^x . e^{-\lambda}}{x!}$ • Mean = Variance = λ

4.6	Normal Distribution	● Understand normal distribution is a Continuous distribution ● Evaluate value of Standard normal variate ● Area relationship between Mean and Standard Deviation	• Characteristics of a normal probability distribution • Total area under the curve = total probability = 1 • Standard Normal Variate: $Z = \frac{x-\mu}{\sigma}$ where x = value of the random variable μ = mean σ = S.D.
UNIT - 5 INFERENTIAL STATISTICS			
5.1	Population and Sample	• Define Population and Sample • Differentiate between population and sample • Define a representative sample from a population • Differentiate between a representative and non-representative sample • Draw a representative sample using simple random sampling • Draw a representative sample using and systematic random sampling	• Population data from census, economic surveys and other contexts from practical life • Examples of drawing more than one sample set from the same population • Examples of representative and non-representative sample • Unbiased and biased sampling • Problems based on random sampling using simple random sampling and systematic random sampling (sample size less than 100)
5.2	Parameter and Statistics and Statistical Interferences	• Define Parameter with reference to Population • Define Statistics with reference to Sample • Explain the relation between Parameter and Statistic • Explain the limitation of Statistic to generalize the estimation for population • Interpret the concept of Statistical Significance and Statistical Inferences • State Central Limit Theorem • Explain the relation between Population-Sampling Distribution-Sample	• Conceptual understanding of Parameter and Statistics • Examples of Parameter and Statistic limited to Mean and Standard deviation only • Examples to highlight limitations of generalizing results from sample to population • Only conceptual understanding of Statistical Significance/Statistical Inferences • Only conceptual understanding of Sampling Distribution through simulation and graphs

5.3	t-Test (one sample t-test and two independent groups t-test)	• Define a hypothesis • Differentiate between Null and Alternate hypothesis • Define and calculate degree of freedom • Test Null hypothesis and make inferences using t-test statistic for one group / two independent groups	• Examples and non-examples of Null and Alternate hypothesis (only non-directional alternate hypothesis) • Framing of Null and Alternate hypothesis • Testing a Null Hypothesis to make Statistical Inferences for small sample size • *(for small sample size: t- test for one group and two independent groups* • Use of t-table
UNIT – 6 INDEX NUMBERS AND TIME BASED DATA			
6.4	Time Series	• Identify time series as chronological data	• Meaning and Definition

6.5	Components of Time Series	• Distinguish between different components of time series	• Secular trend • Seasonal variation • Cyclical variation • Irregular variation
6.6	Time Series analysis for univariate data	• Solve practical problems based on statistical data and Interpret the result	• Fitting a straight line trend and estimating the value
6.7	Secular Trend	• Understand the long term tendency	• The tendency of the variable to increase or decrease over a long period of time
6.8	Methods of Measuring trend	• Demonstrate the techniques of finding trend by different methods	• Moving Average method • Method of Least Squares
UNIT - 7 FINANCIAL MATHEMATICS			
7.1	Perpetuity, Sinking Funds	• Explain the concept of perpetuity and sinking fund • Calculate perpetuity • Differentiate between sinking fund and saving account	• Meaning of Perpetuity and Sinking Fund • Real life examples of sinking fund • Advantages of Sinking Fund • Sinking Fund vs. Savings account
7.3	Calculation of EMI	• Explain the concept of EMI • Calculate EMI using various methods	• Methods to calculate EMI: i) Flat-Rate Method ii) Reducing-Balance Method • Real life examples to calculate EMI of various types of loans, purchase of assets, etc.
7.4	Calculation of Returns, Nominal Rate of Return	• Explain the concept of rate of return and nominal rate of return • Calculate rate of return and nominal rate of return	• Formula for calculation of Rate of Return, Nominal Rate of Return

7.5	Compound Annual Growth Rate	• Understand the concept of Compound Annual Growth Rate • Differentiate between Compound Annual Growth Rate and Annual Growth Rate • Calculate Compound Annual Growth Rate	• Meaning and use of Compound Annual Growth Rate • Formula for Compound Annual Growth Rate
7.7	Linear method of Depreciation	• Define the concept of linear method of Depreciation • Interpret cost, residual value and useful life of an asset from the given information • Calculate depreciation	• Meaning and formula for Linear Method of Depreciation • Advantages and disadvantages of Linear Method
UNIT - 8 LINEAR PROGRAMMING			
8.1	Introduction and related terminology	• Familiarize with terms related to Linear Programming Problem	• Need for framing linear programming problem • Definition of Decision Variable, Constraints, Objective function, Optimization and Non Negative conditions

8.2	Mathematical formulation of Linear Programming Problem	• Formulate Linear Programming Problem	• Set the problem in terms of decision variables, identify the objective function, identify the set of problem constraints, express the problem in terms of inequations
8.3	Different types of Linear Programming Problems	• Identify and formulate different types of LPP	• Formulate various types of LPP's like Manufacturing Problem, Diet Problem, Transportation Problem, etc.
8.4	Graphical method of solution for problems in two variables	• Draw the Graph for a system of linear inequalities involving two variables and to find its solution graphically	• Corner Point Method for the Optimal solution of LPP • Iso-cost/ Iso-profit Method
8.5	Feasible and Infeasible Regions	• Identify feasible, infeasible, bounded and unbounded regions	• Definition and Examples to explain the terms
8.6	Feasible and infeasible solutions, optimal feasible solution	• Understand feasible and infeasible solutions • Find optimal feasible solution	• Problems based on optimization • Examples of finding the solutions by graphical method

UNIT 1: NUMBERS, QUANTIFICATION AND NUMERICAL APPLICATIONS

SOME IMPORTANT RESULTS/CONCEPTS

Modulo Arithmetic:

Euclid ‘s Division Lemma:

For integers a, b($\neq 0$), we have $\boldsymbol{a = bq + r}$, where $q, r \in Z$and $0 \leq r < |b|$

Modulo Arithmetic is the arithmetic of remainders.$\boldsymbol{a mod b = r}$

Where *mod* (modulo) gives the remainder after a is divided by b

Note: 1. $a\ mod\ a = 0$

2.If a<b, then $a\ mod\ b = a$

Properties

1. a **mod b = (a + kb) mod b; where k is any integer**
2.*(A + B)* **mod** *C* **=** *(A* **mod** *C* **+** *B* **mod** *C)* **mod** *C*
3 .(A - B) mod C = (A mod C - B mod C) mod C
4. (A × B) mod C = (A mod C × B mod C) mod C

Congruence Modulo:

Two positive integers *a* and *b* are said to be congruence modulo ***m*** if*a* and *b* satisfy the following conditions:
i) (*a*-*b*) is divisible by *m*
ii) *a* mod *m* = *b* mod *m*

Notation used for congruence modulo is:$a \equiv b(mod m)$

Property 5: If ***a*** ≡***b*** (mod ***m***) where ***a, b*** and ***m*** are positive integers then***ak***≡ ***bk* (mod *m)*** for any positive integer k.

Allegation and Mixture:

$$\text{Ratio}(R) = \frac{\text{C.P.of Dearer (d)} - \text{mean price (m)}}{meanprice(m) - C.Pof cheaper(c)} = \frac{d-m}{m-c}$$

Quantity of Liquid left after n operations $= x(1 - \frac{y}{x})^n$

Where, $x-$ Original amount,$y-$ taken out and $n-$ Number of times

Boats and Streams:

Let the speed of the boat in the still water be $xkm/h$and sped of the stream be ykm/h. Then

1. Downstream Speed $(u) = x + ykm/h$ and Upstream $Speed(v) = x - ykm/h$

2. Speed of Boat $= \frac{u+v}{2}$and Speed of Stream $= \frac{u-v}{2}$

Pipes and Cisterns:

A pipe connected to a tank or cistern which fills it is known as inlet pipe and the pipe connected to the

tank which drains or empties it is known as outlet pipe.

When a tank is connected to many pipes (inlets and outlets), then the difference between the sum of the work done by inlets and the sum of the work

done by outlets gives the filled part of the tank.

- Let a pipe fill a tank in x number of hours, then it can fill (1/x) th portion of the tank in one hour.
- If a pipe can empty a tank in y number of hours, then it can empty out (1/y)th portion of the tank in one hour.
- The portion of tank they can fill together in one hour $(\frac{1}{x}-\frac{1}{y})^{th}$
- If a pipe fills$\frac{1}{x}th$art of a tank in 1 hour, then the time taken by the pipe to fill the tank completely is x hours
- Two pipes can fill a tank in x and y hours respectively. If both the pipes are opened simultaneously, then time taken by both the pipes to fill the tank is $\frac{xy}{x+y}$hours.
- If two pipes A and B together can fill a tank in x hours and the pipe A alone can fill the tank in y hours, then time taken by pipe B alone to fill the tank is$\frac{xy}{y-x}$ hours.
- If a pipe A can fill a tank in x hours and a pipe B can empty the full tank in y hours (where y > x), then net part filled in 1 hour is $\frac{y-x}{xy}$
- If a pipe A can fill a tank in x hours and a pipe B can empty the full tank in y hours (where x >y), then net part emptied in 1 hour $\frac{x-y}{xy}$.
- Three pipes A, B and C can fill a tank in x y, and z hours respectively. If all the three pipes are opened simultaneously, then time taken by all the pipes to fill it is $\frac{xyz}{xy+yz+xz}$hours.
- If two pipes are filing a tank at the rate of x hours and y hours respectively and a third pipe is emptying it at the rate of z hours, then in one hour the part of the tank filled is $\frac{1}{x}+\frac{1}{y}-\frac{1}{z}$ hours

  Time taken to fill the tank is $\frac{xyz}{(yz+zx-xy)}$

Races and Games:

Suppose X and Y are participating in a race.
"X gives Y a start of x meters" means Y starts the race x meters ahead of X.
"X gives Y a start of t-minutes" means X will start t minutes after B starts the race.
"X beats Y by x meters" means when X reach the finishing point, Y is x meters behind X.
"X beats Y by t minutes" means Y finish t minutes after X.

"X beats Y by x meters and t minutes" means Y is x-meters behind X and finish race t-minutes after X.

A game of 100 means that the person among the participants who scores 100 points first is the winner. "X beats Y by 25 points" or "X can give Y 25 points" means X scores 100 while Y scores only 75points (100-25).

Numerical Inequalities:

Any two real number associated by, $\leq or \geq$ forma numerical inequality If a, b are positive numbers and AM and GM are their arithmetic mean and geometric mean respectively,

then $AM = \frac{a+b}{2} and \boldsymbol{GM} = \sqrt{ab}$

$$AM - GM = \frac{a+b}{2} - \sqrt{ab}$$

$$= \frac{a + b - 2\sqrt{ab}}{2}$$

$$= \frac{(\sqrt{a} - \sqrt{b})^2}{2} \geq 0$$

$\boldsymbol{AM \geq GM}$

MULTIPLE CHOICE QUESTIONS

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1 A man can row 7.5 km/hr in still water. if the stream is flowing at the rate of 1.5 km/hr, it takes him 50mins to row to a place and return . how far is the place?

a. 3km   b. 2km   c. 4km   d.2.5 km

Ans: a. 3km

Solution: $U = 7.5km/hr , V = 1.5km/hr$

Let d be the distance.

Total time = time in upstream + time in downstream

$\frac{50}{60} = \frac{d}{U+V} + \frac{d}{U-V}$ ⇒d=3km

2	$[(3\times 7)+5]mod4$ is a.3 b.2 c.4 d.5 Ans: b.2 **Solution:** $[(3\times 7)+5]mod4 = 26mod4$ $26 = 6\times 4+2 \Rightarrow 26\equiv 2\ (mod4)\ =\ 2$
3	$x\equiv 4(mod7)$ then positive values of x are a. $\{4,11,18 \ldots.\}$ b. $\{11,18,25 \ldots.\}$ c. $\{4,8,12 \ldots.\}$ d. $\{1,8,15 \ldots.\}$ Ans : a. $\{4,11,18 \ldots.\}$ **Solution:** $\mathrm{x}\equiv 4(\mathrm{mod}7)\Rightarrow \mathrm{x}=7\mathrm{k}+4\,,\quad \mathrm{k}=0,1,2\ldots$
4	Milk and water in two vessels are in the ratio 5:3 and 5:4 respectively .in what ratio liquid in both vessels be mixed to obtain a new mixture in which ratio of milk and water is 7:5 respectively . a. 3:2 b.3:5 c. 2:3 d. 2:5 Ans : c. 2:3 **Solution:** $\frac{7}{12}-\frac{5}{9}\ :\frac{5}{8}-\frac{7}{12}\ \Rightarrow \frac{1}{36}:\frac{1}{24}\Rightarrow\ 2:3$
5	In what ratio must rice at Rs 69 per kg be mixed with rice at Rs 100 per kg so that the mixture be worth Rs 80 per kg? a . 11:20 b. 11: 10 c. 20:11 d. 10:11 Ans: c. 20:11 **Solution:** Rice at 69 : rice at 100 = $100-80 : 80-69\ =\ 20:11$
6	In a 50m race A can give a start of 5m to B and a start of 14 m to C .In the same race how much start can B give to C? a. 9m b.10 m c. 12m .d.11 m Ans: b.10 m **Solution**: $B:A=50:45\qquad A:C=50:36$ $\frac{B}{C}=\frac{B}{A}\times\frac{A}{C}=\frac{45}{36}=\frac{50}{x}\Longrightarrow x=40$ B can give a start of C by$50-40=10m$

7 A runs $1\frac{2}{3}$ times as fast as B. If A gives B a start of 80 m, how far is the running post so that A and B may reach it at the same time .

a. 100m b.120m c.200 m d.240 m

Ans: c.200 m

Solution: $A: B = 1\frac{2}{3}: 1$

$$A: B = \frac{5}{3}: 1$$
$$A: B = 5: 3$$

Let $A = 5xB = 3x$

A gives B a start of 80 m ,$2x = 80x = 40$

Distance = $5 \times 40 = 200$

8 pipes A and B together can fill a pipe in 4 hours ,pipe B take 6 hours more than A to fill the tank ,if they opened separately .The time taken by A to fill the tank alone is

a. 2 hours b. 4 hours c. 6hours d. 8 hours

Ans: c. 6hours

Solution:

Let Time taken by A = $xhrs$ and B $= (x + 6)hrs$

Time taken by A and B together = 4 hours

In 1 hour they can fill $\frac{1}{4}$ part of the tank

$$\frac{1}{x} + \frac{1}{x + 6} = \frac{1}{4}$$
$$4(2x + 6) = x^2 + 6x$$
$$x^2 - 2x - 24 = 0$$
$$x = 6or4$$
$$x = 6hours$$

9 -41 mod7 is ……

a. -6 b. 5 .c. 1 d. -1

Ans: .c. 1

Solution: $-41 = 7 \times -6 + 1$

$-41mod7 = 1$

10 A person can row a boat 5 km an hour in still water. It takes him thrice as long to row upstream as to row downstream. Find the rate at which the stream is flowing.

a. 1.5 km/hr b.2.5 km/hr c. 2km/hr d. 3 km/hr
Ans : b. 2.5 km/hr

Solution: Let the rate at which the stream is flowing be x km/hrand let the distance covered by the boat be y km/he

$$\frac{3y}{5+x} = \frac{y}{5-x}$$

$$15 - 3x = 5 + x$$
$$x = 2.5hrs$$

11 A cistern can be filled by two pipes *A* and *B* in 12 minutes and 15 minutes respectively. Another tap *C* can empty the full tank in 20 minutes. If the tap *C* is opened 5 minutes after the pipes *A* and *B* are opened, find when the cistern becomes full?

a.8min 30 sec b. 8 min 20 sec c. 7min 30 sec d. 7 min 20 sec

Ans:7 min 30 sec

12 In a 1000 meters race. *A*, *B*, and *C* get the gold, silver, and bronze medals, respectively.
If *A* beats *B* by 100 meters and *B* beats *C* by 100 meters, then by how many meters does *A* beat *C*

a. 200m b. 180 m c. 170m d. 190m
Ans: a. 200m

13 Two runners *A* and *B* complete a 100 meters race in 36seconds and 48 seconds respectively, by how many meters will *A* defeat *B*

a. 25m b. 20 m c. 30m d. 15m

Ans : 25 m

14 if a= 11 b=43, then subtraction modulo 7 is…

a. 3 b.-3 c.4 d.6

Ans : 3

Assertion and Reason Type questions

The following questions consist of two statements, one labelled as 'Assertion (A)' and the other labelled as 'Reason (R)'. You are to examine these two statements carefully and decide if the Assertion (A) and Reason (R) are individually true and if so, whether the Reason (R) is the correct explanation for the given Assertion (A). Select your answer to these items using the codes given below and then select the correct option.

Codes:

A. Both A and R are individually true and R is the correct explanation of A

B. Both A and R are individually true but R is not the correct explanation of A

C. A is true but R is false

D. A is false but R is true

1	**Assertion A** :$(486+729)mod\ 12 \equiv 3$ **Reason R**: $(a+b)modn = a(modn) + (b\ mod\ n)$ $(486+729)mod\ 12 = 1215mod12 = 101 \times 12 + 3$ $486\ mod\ 12 = 6 \quad 729\ mod\ 12 = 9$: $486\ mod\ 12 + 729\ mod\ 12 = 15\ mod\ 12 = 3$ Ans :A) Both Assertion and Reason are true and R is the correct explanation of A
2	**Assertion A**:$186 \times 93\ mod\ 7 \equiv 2$ **Reason R**:$a.b(mod\ n) = (a\ modn).(b\ modn)$ Ans :D) A is false but R is true
3	**Assertion A**: The ratio of copper and Zinc in brass is 13:7. In 100 kg of Brass there is 35 kg of Zinc **Reason R**: Ratio = $\frac{cp\ \ of\ dearer - mean\ price}{mean\ price - cp\ of\ cheaper}$ Amount of Zinc in 100 kg Brass= $100 \times \frac{7}{20} = 35\ kg$ Ans : B) A and R are true but R is not the correct explanation of A

4	**Assertion A**: Rohan can row with a speed of 16 km /hr in still water. If the speed of stream is 12km/hr ,then speed of Rohan with stream will be 26 km/hr **Reason R**: If the speed of a boat in still is xkm/hr and speed of the stream is ykm/hr, then speed of down- stream will be x+y km/hr Ans: $x = 16km/hr \quad y = 12\ km/hr$ $x + y = 28\ km/hr$ Ans :D) Assertion is False Reason is True
5	**Assertion A**: If a is any positive real number then $a + \frac{1}{a} \geq 2$ **Reason R**: Let a and b be distinct positive real numbers then $\frac{a+b}{2} > \sqrt{ab}$ $ie, AM > GM$ Ans : A) A and R are true and R is the correct explanation of A
6	**Assertion A**: Two pipes A and B can fill a tank in 20 hrs and 30 hrs respectively.If both the pipes open simultaneously, then pipe A should be closed after 8 hrs so that the tank is filled in 18 hrs. **Reason R** : Time taken to fill the tank is positive and the time taken to empty a tank is taken negative. Ans : B) A and R are true but R is not the correct explanation of A
7	**Assertion A**: In a 50 m race, A can give a start of 5m to B and a start of 14 m to C ,the B gives a start of 40 m to C **Reason R**: A gives a start of x m means that before the start of race, A is at the starting point and B is ahead of A by x m, B give a start of 10 m to C . Ans :D) Assertion is False Reason is True
8	**Assertion A** :It is currently 8 am .It will be 4 am in next 500 hrs. **Reason R**: We know that time repeats in every 24 hours . So ,$500\ mod\ 24 = 20$ Therefore 500hrs is equivalent to 20 hrs. Ans :B)Both A and R are true but R is not the correct explanation of A

9	**Assertion A**: Let x and b are real numbers ,if $b > o\ and\ \lvert x\rvert > b$, then $x\epsilon(-\infty,-b) \cup (b,\infty)$ **Reason R**:Let x and b are real numbers ,if $b > o\ and\ \lvert x\rvert > b$, then $x\ \epsilon\ [-\infty,b] \cup [b,\infty]$ Ans :C)A is true and R is false
10	**Assertion A**: The last two digits of the product 2345×6789 is 05 **Reason R**:To find the last two digits of the product 2345×6789, we find $2345 \times 6789(mod\ 100)$ Ans :A) Both A and R are true but R is the correct explanation of A
11	**Assertion A**: $3^{16}(mod4) \equiv 1$ **Reason R :** if $a \equiv b(mod\ n)$,then$a^k \equiv b^k(mod\ n), \forall\ k\epsilon N$ Ans :A) Both A and R are true but R is the correct explanation of A
12	**Assertion A**: In a 1000 m race , A can beat B by 100m and in a 400m race ,B can beat C by 40 m ,in a 500m race A can beat C by 95 m **Reason R**:A beats C by of x m means C is behind A by x m ,when A reached of finishing point. Ans :A) Both A and R are true but R is the correct explanation of A
13	**Assertion A**: If a boat takes 12 hours to row 48 km upstream and 8hours to row the same distance down stream ,then the speed of boat in still water is 5 km/hr and speed of river is 1km/hr. **Reason R**:Let speed of boat and speed of river is x km/hr and y km/hr respectively, Then downstream speed$= x - y\ km/hr$ Upstream speed $= x + y\ km/hr$ Also downstream speed $= \frac{distance\ covered}{time\ taken}$ Up stream speed$= \frac{distance\ covered}{time\ taken}$ Ans :A) Both A and R are true but R is the correct explanation of A

14 **Assertion A:** when 14 is divided by 5 ,the remainder is 4.So we write $14\ mod\ 5 = 4$

Reason R: $37 \equiv 12(mod\ 5)$

Here $37 - 12 = 25$ is an integral multiple of 5

Ans:D) A is false but R is true

15 **Assertion A:**The ratio in which water must be added to dilute honey costing Rs 240 per litre so that the resulted syrup would be worth Rs 200 per litre is 1:4

Reason R : Ratio = $\frac{cp\ \ of\ dearer - mean\ price}{mean\ price - cp\ of\ cheaper}$

Ans: B) Both A and R are individually true but R is not the correct explanation of A

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

1 Find the remainder when 987+876+765+654+543+432+321+210 is divided by 6.

Solution:
987 ≡ 3 (mod6)
876 ≡ 0 (mod6)
765 ≡ 3 (mod6)
654 ≡ 0 (mod6)
543 ≡ 3 (mod6)
432 ≡ 0 (mod6)
321 ≡ 3 (mod6)
210 ≡ 0 (mod6)
(987+876+765+654+543+432+321+210)mod 6
=(1+0+1+0+1+0+1+0)mod6
=4(mod6)
=2.

2 It is 7:00 P.M. currently. What time (in A.M.or P.M.) will be in the next 1500 hours?

Solution:
Time repeats after every 24 hours.
= 1500 (mod 24)
= 12
(mod24)
therefore,
1500hrs =
12hrs
So, Time = 7:00 AM

3	In what ratio does a grocer mix two varieties of pulses worth Rs.85 per kg and Rs.100 per kg respectively so as get a mixture worth Rs.92 per kg? **Solution:** C.P of cheaper pulse ‘c’ = Rs.85 per kg C.P of dearer pulse ‘d’ = Rs.100 per dayMixture ‘m’ = Rs.92 per kg Quantity of c/Quantity of d = (d-m)/(m-c) = (100-92)/(92-85) = 8/7 Ratio = 8:7
4	A woman can swim 8 km/h in still water. If the speed of the stream is 4 km/h, thenfind the time taken by the woman to cover the distance of 16 km upstream. **Solution:** Speed of still water (x) = 8 km/hr Speed of the stream (y) = 4 km/hr Distance Covered (d) = 16 km Speed of upstream (v) = x – y = 8 – 4 = 4 km/hr Time Taken in upstream = d/s = 16/4 = 4 hour
5	Two pipes can fill the tank in 20minutes and 24 minutes respectively and a waste pipe can empty3 gallons of water per minute If all the three pipes working together can fill the tank in 15 minutes, find the capacity of the tank? **Solution:** $\frac{1}{20}+\frac{1}{24}-\frac{1}{t}=\frac{1}{15}$ t=40 Volume of water emptied in one minute = 30gallons So Volume of water emptied in 40 minutes= 120gallons Capacity of tank = 120 gallons.
6	A dealer has 1000 kg sugar and he sells a part of it at 8% profit and the rest of it at 18% profit. The overall profit he earns is 14%. What is the quantity which is sold at18% profit? **Solution:** Quantity of Dearer: Quantity of Cheaper = (18-14) : (14-8) = 4:6 = 2:3Quantity of sugar sold at 18% profit = 3/5 × 1000 = 600kg
7	A and B run a km and A wins by 1 minute. A and C run a km and ‘A’ wins by 375 meters. B and C run a km and B wins by 30 seconds. Find the time taken by each to run a km.

Solution:

Since A beats B by 60 seconds and B beats C by 30 seconds. So, A beats C by 90 seconds. But, it being given that A beats C by 375 meters. So it means that C covers 375 meters in 90 seconds.

∴ Time taken by C to cover 1 km

$= \frac{90}{375} \times 1000$ seconds

= 240 seconds

Time taken by A to cover 1 km

= (240-90) seconds

= 150 seconds

Time taken by B to cover 1 km

= (240-30) seconds

= 210 seconds.

8 Solve the inequality for real x:

$4 - 2x \geq 3x + 19$

Solution:

$\boldsymbol{x} \leq \mathbf{-3}$ ***OR*** x $\in (-\infty, \mathbf{-3}]$

9 A # B, means A is greater than B,
A * B, means A is smaller than B,
A % B, means A is equal to B
A @ B, means A is greater than equal to B,
A © B, means A is smaller than equal to B
Statement: S © P @ Q # R
Conclusion I: S @ R
Conclusion II: R * P

1. Only conclusion I is true
2. Only conclusion II is true
3. Both conclusion I and II are true
4. Neither conclusion I nor II is true
5. Either conclusion I or II is true

Ans:2

10 Find all pairs of consecutive positive integers, both of which are larger than 5, such that their sum is less than 23

Solution:

To find all pairs of consecutive positive integers, both of which are larger than 5, and whose sum is less than 23, you can set up the following inequality:

x + (x + 1) < 23

Here, x represents the first integer, and x + 1 represents the consecutive integer.

Now, solve the inequality:

2x + 1 < 23

Subtract 1 from both sides:

2x < 22

Divide by 2:

x < 11

Since both integers need to be larger than 5, you also need to consider this condition:

x > 5

Now, you have two conditions for the integers:

5 < x < 11

You can list all pairs of consecutive positive integers that satisfy these conditions:

1. (6, 7)
2. (7, 8)
3. (8, 9)
4. (9, 10)

These are all the pairs of consecutive positive integers, both of which are larger than 5, and their sum is less than 23.

SHORT ANSWER TYPE QUESTIONS (3 MARKS)

1	Find the last two digits of the product 4895×6789. **Solution:** = 4895×6789 (mod 100) = 4895 (mod 100) x 6789 (mod 100) = 95(mod 100) x 89(mod 100) = 95 x 89 (mod 100) = 8455 (mod 100) = 55 (mod 100)
2	What is the remainder when 17^{113} is divided by 3? **Solution:** 17 ≡ −1 (mod 3) $17^{113} \equiv (-1)^{113}$ (mod3) ≡ −1 (mod 3) ∴17^{113} has remainder 2 when divided by 3.
3	Two vessels P and Q contain milk and water in the ratio 5:3 and 13: 3 respectively. In what ratio mixtures from two vessels should be mixed to get a new mixturecontaining milk and water in the ratio 3 : 1 respectively? **Solution:** Quantity of milk in vessel P = 5/8 Quantity of milk in vessel Q = 13/16Quantity of mixture of milk = ¾ L.C.M = 8,16,4 = 16 Therefore, Quantity of milk in vessel P = 10/16 (C) Quantity of milk in vessel Q= 13/16 (d) Quantity of milk in mixture = 12/16(m) (d-m)/(m-c) = (13/16-12/16)/(12/16-10/16) = 1/16/2/16 = ½ Required Ratio = 1:2
4	A boat goes 30 km downstream and comes back to the starting point in 4 hours and 30 minutes. If the speed of the boat in still water is 15 km/h, find the speed of the stream. **Solution:** Let's denote the speed of the stream as 's' km/h. Downstream speed = (speed in still water) + (speed of the stream) = 15 + s km/h Upstream speed = (speed in still water) - (speed of the stream) = 15 - s km/h

	Time taken for downstream journey = distance/speed = 30/(15+s) hours Time taken for upstream journey = distance/speed = 30/(15-s) hours Total time taken = 4 hours 30 minutes = 4.5 hours 4.5 = 30/(15+s) + 30/(15-s) Solving for 's', we get s = 5 km/h Therefore, the speed of the stream is 5 km/h
5	Tea worth Rs.126 per kg and Rs135perKg are mixed with a third variety in the ratio 1:1:2. If the mixture is worth Rs 153 per Kg, find the price of the third variety in per Kg. **Solution:** Mixture price = Rs 153 $\frac{126\ x\ 1+135\ x1+2x}{1+1+2}$ =153 X = $\frac{351}{2}$
6	A and B can cover a 200 m race in 22 seconds and 25 seconds respectively. When A finished the race, then B is at what distance from the finishing line? **Solution:** Distance covered by B in 25 sec. = 200 m. Distance covered by B in 22 sec. = ($\frac{200}{25}$ x22)m =176m ∴ B was at a distance of (200 – 176) m = 24 m from the finishing line
7	$If\ a, b, c\ are\ positive\ real\ numbers, then\ find\ the\ least\ value\ \ of$ (a+b)(b+c)(c+a) **Solution:** AM≥ GM $(a + b)(b + c)(c + a) \geq 8abc$
8	There are 3 points P, Q and R in a straight line such that Q is equidistant from P and R . A can swim from P to R downstream in 24 hours and from Q to P upstream in 16 hours. Find the ratio of speed of man in still water to speed of stream? **Solution:** Let speed of man in still water = x km/h Speed of current = y km/h Downstream speed = (x + y) km/h

	Upstream speed = (x – y) km/h Let PQ = QR = A and PR = 2A $\frac{2A}{x+y}$ =24, $\frac{A}{x-y}$ = 16 By dividing both equations ⇒ 4x – 4y = 3x + 3y ⇒ $\frac{x}{y} = \frac{7}{1}$ Required ratio = Speed of man in still water : Speed of current ⇒ 7 : 1
9	Find -8(mod5) **Solution:** -8(mod5) = 2 (remainder is positive)

EXERCISE (2MARKS)

1	There are 81 boxes with 21 articles in each. When we rearrange all of the articles so that each box has 5 articles, how many articles will be left out without a box? **Ans:** 1
2	A can run a km in 3 min. 10 sec. and B in 3 min. 20 sec. By what distance can A beat B ? **Ans:** 50 m
3	A boat goes 360 km upstream and returns to the same point in 35 hours. If the speedof current is 3km/h, how much distance boat will cover in still water in 6 hours? **Ans:** 21 x 6 = 126 KM
4	In a game of 100 points, A can give B 20 points and C 28 points. Then, Bcan give C. **Ans:** 10 points
5	A vessel is filled with liquid, 3 parts of which are water and 5 parts syrup. How much of the mixture must be drawn off and replaced with water so that the mixture may be half water and half syrup? **Ans:** 1/5

EXERCISE (3MARKS)

1	Find the last digit of 17^{17} **Ans:** 7
2	In what ratio must a grocer mix rice worth Rs. 30 per kg and worth Rs. 32.5 per kg ,so that by selling the mixture at Rs. 34.10 per kg, he may gain 10%? **Ans:** 3:2
3	A cistern can be filled by two taps A and B in 12 hours and 16 hours respectively. The full cistern can be emptied by a third tap C in 8 hours. If all the taps are turned on at the same time, in how much time will the empty cistern be filled completely? **Ans:** 48 hrs
4	If the time after 640hrs from now will be 9a.m, then what is current time? **Ans:** 5.pm
5	How many litres of 25% solution of acid, should be added to 600 litres of 10% solution of acid so that the resulting mixture will contain more than 12% but lessthan 15% of acid content? **Ans:** 1200/13< x<300

LONG ANSWER TYPE QUESTIONS

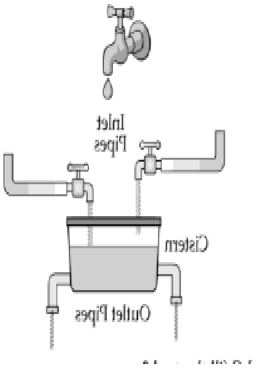
1	Two pipes can fill a cistern in 8 and 12 hours respectively.The pipes are opened simultaneously and it takes 12 minutes more to fill the cistern due to leakage. If the cistern is full what will be the time taken by the leakage to empty it **Solution:** In one hour pipe A can fill 1/8th part of the tank and pipe can fill 1/12th part of the tank Total part of the tank filled by both A and B (opened simultaneously) in one hour=$\frac{1}{8}+\frac{1}{12}=\frac{5}{24}$ Time taken by both A and B =$\frac{24}{5}$ =$4\frac{4}{5}$hrs

	Time taken by A and B due to leakage=$\frac{24}{5}+\frac{12}{60}=\frac{24}{5}+\frac{1}{5}=\frac{25}{5}=5\,hrs$ Let time taken by leakage to empty the tank be x hrs $\frac{1}{8}+\frac{1}{12}-\frac{1}{x}=\frac{1}{5}$ On solving x=120hrs
2	A steamer can go 24 km in still water in 50 minutes. One day, it went 22.5 km upstream and returned the same distance in downstream. If the difference between the time taken to travel upstream and downstream was 25 minutes, then what was the speed of stream in km per hour? **Solution:** Speed of steamer in still water=24000/50=480m/minute=8m/sec Let speed of stream=v m/sec In upstream ,the speed of steamer=(8-v) m/ses In downstream the speed of steamer=(8+v) m/sec 22500/(8-v) - 22500/(8+v) = 25 × 60 22500/(8-v) - 22500/(8+v) = 1500 v =2m/sec $=2\times\frac{18}{5}\frac{km}{hr}$ =7.2km/hr
3	A milk man bought 15 litres of milk and mixed 3 lires of water in it.If the price per litre of the mixture becomes Rs.22, what is cost price of the milk per litre? **Solution:** Let cost price of milk be x per kg Price of 15kg of milk = 15x After mixing 3 kg of water quantity of mixture = (15+3) kg = 18kg Then the price of the mixture is Rs.22 per kg

	$15x = 22\times 18 = 132$ $x = \frac{132}{5} = 26.40$
4	A cistern has three pipes A B, and C . A and B can fill it in 3 hours and 4 hours respectively while C can empty the completely filled cistern in 1 hour. If the pipes are opened in order at 3 P.M., 4 P.M. and 5 P.M. respectively, at what time will the cistern be empty ? (Ans 7 : 12 pm)
5	A man rows to a place 46 km distance and back in 11 hours 30 minutes. He found that he can row 5 km with the stream in the same time as he can row 4 km against the stream. Find the rate of the stream. (Ans 0.9 km/h)
6	Tea worth Rs. 126 per kg and Rs. 135 per kg are mixed with a third variety in the ratio 1:1 :2. If the mixture is worth Rs. 153 per kg, the price of the third variety per kg will be: (Ans Rs.175.50)

CASE BASED QUESTIONS:

1. An overhead water tank has three pipes A,B and C attached to it . The inlet pipes A and B can fill the empty tank independently in 15 hours respectively. The outlet pipe C alone can empty a full tank in 20 hours.



Based on the above information answer the following questions

(a) For a routine cleaning of the tank, the tank needs to be emptied. If pipes A and B are closed at the time when the tank is filled to two-fifth of its total capacity. How long will pipe C take to empty the tank completely

(b) How long will it take for the empty tank to fill completely if all the three pipes are opened simultaneously

(c) On a given day, pipes A,B and C are opened (in order) at 5am,8am,and 9am respectively to fill the empty tank, in how many hours will the tank be filled completely

OR

Given that the tank is half-full. only pipe C is opened at 6am, to empty the tank. After closing the pipe C and an hour's cleaning time ,the tank is filled completely by pipe A and B together, what is the total time taken in the whole process.

Solution:

(a) Time taken by Pipe C to empty one full tank is 20 hours

Time taken by Pipe C to empty two fifth tank is$\frac{2}{5} \times 20 = 8\ hours$

(b) Time taken for the empty tank to fill completely if all the three pipes are opened

$= \frac{1}{15} + \frac{1}{12} - \frac{1}{20} = \frac{1}{10}$

(c)$\frac{x}{15} + \frac{x-3}{12} - \frac{x-4}{20} = 1$ (pipe B opened 3 hrs later than A and pipe C opened 4hrs later than A)

4x+5x-15-3x+12=6

6x=63

x=$10\frac{1}{2}$ hours

OR

Time taken by C to empty the tank=10hours

Cleaning time =1 hour

Part of time filled by A and B in 1 hour

$=\frac{1}{15}+\frac{1}{12}=\frac{3}{20}$

Time taken=$\frac{20}{3}=6\frac{2}{3}hours = 6\ hours\ 40\ minutes$

Total time= 10+1+6hours 40 minutes = 17 hours 40 minute

2. Seema is rowing a boat. She takes 6 hours to row 48 km upstream where she takes3hours to go downstream.

Based on the above information answer the following questions

(a) What is her speed of rowing in still water
(b) What is her average speed
(c) The stream is flowing at the speed of 4km/hr. If she rows a certain distance upstream in 3.5hrs and returns to the same place in 1.5 hrs then what is the speed of seema's boat in still water

Solution:

Upstream speed V=$\frac{48}{6}$ =8km/hr

Downstream speed u=$\frac{48}{3}$ =16km/hr

(a) X=$\frac{u+v}{2}=\frac{24}{2}$ =12km/hr
(b) Y=$\frac{u-v}{2}=\frac{8}{2}$ =4km/hr

Average speed=$\frac{x^2-y^2}{x}=\frac{128}{12}=10\frac{2}{3}$ km/hr

(c) y=4km/hr

t_2=3.5hr t_1=1.5hr

$$\frac{x}{y}=\frac{t_2+t_1}{t_2-t_1}$$

$\frac{x}{4} = \frac{3.5+1.5}{3.5-1.5} = \frac{5}{2}$

x=10km/hr

3. In mathematics modular arithmetic is a system of arithmetic for integers where numbers ”wraparound” when reaching a certain value called modulus . A familiar use of modular arithmetic is in the 12 hour clock in which the day is divided into two 12 hour periods. If the time is 7:00 now, then 8 hours later it will be 3:00. Simple addition would result in 7+8=15,but clocks “wrap around” every 12 hours. Because the hour number starts over after it reaches 12, this is arithmetic modulo 12. In terms of the definition,15 is congruent to 3 modulo 12. So 15:00 on a 24 hour clock is displayed 3:00 on a 12 hour clock. Based on the above information answer the following questions

(a) Evaluate 3^6(mod4)

(b) What is the least positive of x for which 100$\equiv x(mod7)$

(c) Evaluate (137+995)mod 12

OR

Find the last digit of 12^{12}

Solution:

(a)$3^2 \equiv 1(mod4)$

($3^2)^3 \equiv 1^3$ (mod4)

$3^6 \equiv 1(mod\ 4)$

$3^6 mod\ 4 = 1$

(b)100$\equiv x(mod7)$

100-x is divisible by 7 when x=2,100-2=98 which is divisible by 7

Therefore the least positive value of x=2

(c)(137+995)mod 12

= (5+11)mod 12=16mod 12=4

OR

$12^{12} mod 10$

12 mod 10=2

$12 \equiv 2\ mod 10 12^6 \equiv 2^6\ (mod 10)$

=64(mod10)=4

$(12^6)^2 = 4^2 = 16\ mod\ 10 = 6$

Therefore last digit =6

4. There are 24 hours in a day. To know the time we use clocks. There are mainly two types of clocks. 12 hours clock and 24 hours clock. 12 hours clock repeat itself twicein a day. That is 24 hours of a day are divided into periods called am(antemeridian) and pm(post meridian). Each period consists of 12 hours numbered 12 (acting as 0) 1,2,3,4,5,6,7,8,9,10,11. Then 24 hours per day cycle starts at 12 midnight(usually indicated as 12 pm) and continues just before midnight at the end of the day,Based on the above information answer the following questions

(a) It is currently 7:00am in 12hours clock,what will be the time in next 492 hours(Ans 7:00pm)
(b) If the time after 640 hours from now will be 9:00pm then what is current time(Ans 5:00am)
(c) What time in 12 hours clock is equivalent to 20:00 in 24 hours clock(Ans 8:00pm)

Q2.For providing water to the families of a colony a large water tank with two inlet pipes A and B and an outlet pipe C is installed.
Pipes A and B can fill the tank in 10 hrs and 12 hrs respectively where as the pipe C can empty the tank in 15 hours. Based on the above information answer the following questions

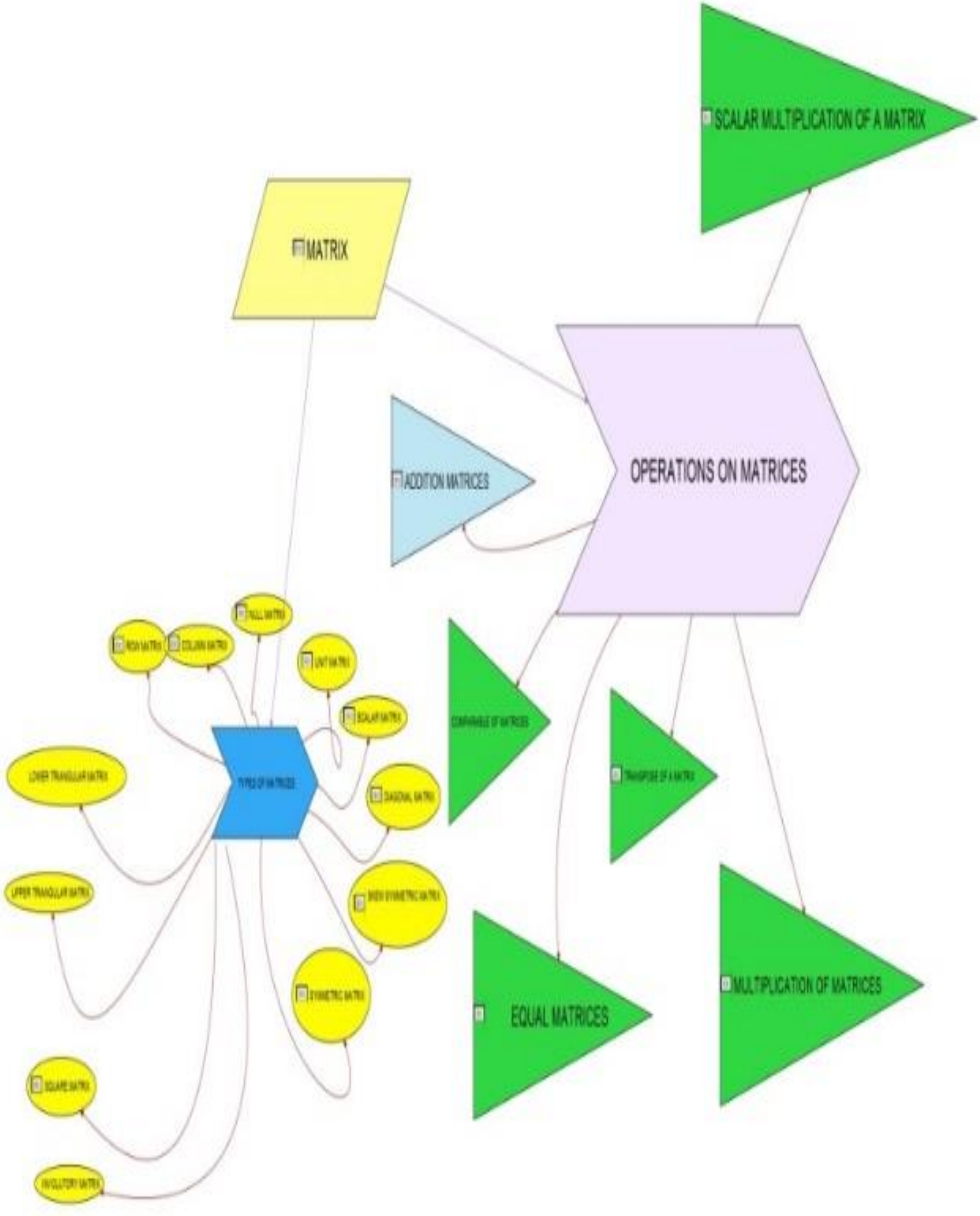
(a) If both the pipes A and B are opened together then in how much time the tank will be filled(Ans: $5\frac{5}{11}$hrs)
(b) If both pipes A and C are opened together then in how much time the tank will be filled(Ans: 30hrs)
(c) If all the pipes are opened together then in how much time the tank will be filled(Ans:$8\frac{4}{7}\, hrs$)

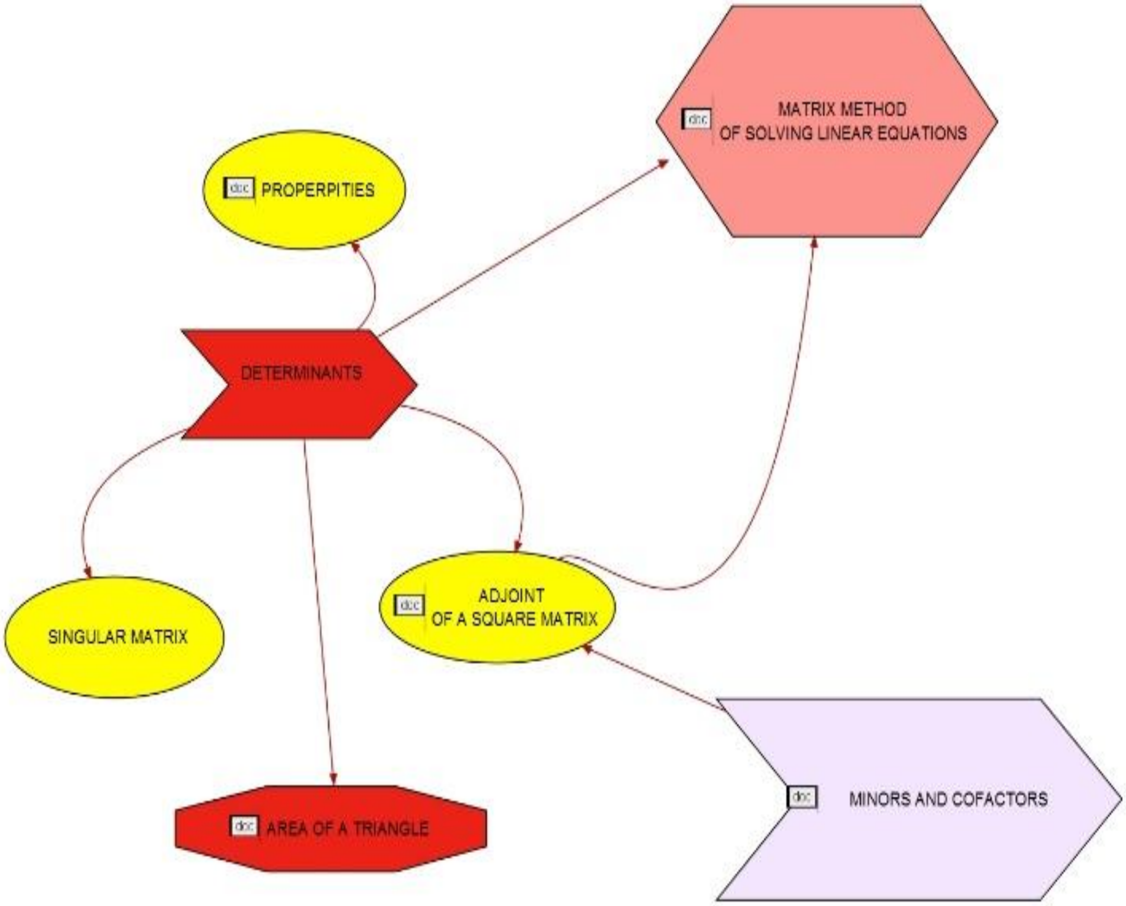
OR

If both the pipes A and B are opened together for some time and then pipe B is turned off. If the tank is filled in hours then after how many hours the pipe is turned off.(Ans: 4.8 hrs)

UNIT 2: ALGEBRA (MATRICES AND DETERMINANTS)

Concept mapping





LEARNING OUT COMES:

1. Define matrix
2. Identify different kinds of matrices
3. Find the size/order of matrices.
4. Determine equality of two matrices
5. Write the transpose of matrices
6. Define symmetric and skew symmetric
7. Perform operations like addition and subtraction of matrices of same order
8. Perform multiplication of two matrices of appropriate order
9. Find determinant of a square matrix
10. Define the inverse of square matrix.
11. Solve the system of simultaneous equations using (i) Cramer's rule (ii) inverse of coefficient matrix.

MATRICES AND DETERMINANTS

DEFINITIONS

A matrix is a rectangular array of m $\times$n numbers arranged in m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots\ldots a_{2n} \\ \\ a_{m1} & a_{m2}\ldots\ldots\ldots a_{mn} \end{bmatrix}_{m\times n} \qquad \text{OR } A = \left[a_{ij}\right]_{m\times n}, \text{ where } \mathbf{i} = 1, 2,., m\ ;\ \mathbf{j} = 1, 2,\ldots.,n.$$

Note :1. Element a_{ij} =($a_{i^{th}row\ j^{th}column}$) ie *which lies in* i^{th}*row* j^{th}*column of matric*

$$\begin{array}{cccc} \begin{array}{c} \\ \\ \\ \end{array} \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] & \begin{array}{l} \rightarrow row\ 1 \\ \rightarrow row2 \\ \rightarrow row3 \end{array} \end{array}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ column1 & column\ 2 & column\ 3 \end{array}$$

*** Order of Matrix**: Number of Row × Number of columns

*** Row Matrix**: A matrix which has one row is called row matrix. A=$\left[a_{ij}\right]_{1\times n}$

For example, $B = \begin{bmatrix} -\frac{1}{2}\sqrt{5} & 2 & 3 \end{bmatrix}_{1\times 4}$ is a row matrix.

*** Column Matrix**: A matrix which has one column is called column matrix. A =$\left[a_{ij}\right]_{m\times 1}$.

For example, $A = \begin{bmatrix} 0 \\ \sqrt{3} \\ -1 \\ \frac{1}{2} \end{bmatrix}$ is a column matrix of order 4 × 1.

*** Square Matrix**: An $m \times n$ matrix is said to be a square matrix if $m = n$ (ie no. of Row is equal to no. of column)and is known as a square matrix of order ' n '. A =$\left[a_{ij}\right]_{n\times n}$.

For example $A = \begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ is a square matrix of order 3.

*** Diagonal Matrix**: A square matrix is called a Diagonal Matrix if all the elements, except the diagonal elements are zero. A = $\left[a_{ij}\right]_{n\times n}$, where$a_{ij}$= 0 , **i** ≠ j and a_{ij}≠ 0 , **i** = j.

*** Scalar Matrix**: A square matrix is called scalar matrix it all the elements, except diagonal elements are zero and diagonal elements are same non-zero quantity.

A = $\left[a_{ij}\right]_{n\times n}$, where a_{ij}= 0 , **i** ≠ j. a_{ij} =α , **i** = j.

*** Identity Matrix**: A square matrix in which all the non diagonal elements are zero and diagonal elements are unity is called identity or unit matrix.

*** Null Matrices**: A matrices in which all element are zero.

*** Equal Matrices**: Two matrices are said to be equal if they have same order and all their Corresponding elements are equal.

***Multiplication of a matrix by a scalar:** if A = $\left[a_{ij}\right]_{m\times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k.

Example: A=$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$ then k A = $\begin{bmatrix} kx_1 & ky_1 & kz_1 \\ kx_2 & ky_2 & kz_2 \\ kx_3 & ky_3 & kz_3 \end{bmatrix}$

***Product of matrices:**

(i) If A & B are two matrices, then product AB is defined, if Number of column of A = number of rows of B.

i.e. A = $\left[a_{ij}\right]_{m\times n}$, B = $\left[b_{jk}\right]_{n\times p}$ then AB =$[c_{ik}]_{m\times p}$.

(ii) Product of matrices is not commutative. i.e. AB ≠ BA.

(iii) Product of matrices is associative. i.e A(BC) = (AB)C

(iv) Product of matrices is distributive over addition. A(B+ C) = AB +AC

* **Transpose of matrix**: If A is the given matrix, then the matrix obtained by interchanging the rows and columns is called the transpose of a matrix. Denoted by A^T

Properties of Transpose:

If A & B are matrices such that their sum & product are defined, then

(i) $(A^T)^T$ =A (ii) $(A+B)^T = A^T + B^T$

(iii) $(KA^T) = KA^T$ where K is a scalar. (iv) $(AB)^T = B^T A^T$

(v) $(ABC)^T = C^T B^T A^T$.

Symmetric Matrix: A square matrix is said to be symmetric if A = A^T

i.e. If A = $\left[a_{ij}\right]_{m\times m}$ then $a_{ij}= a_{ji}$ for all i, j. Also elements of the symmetric matrix are symmetric about the main diagonal

For Example A=$\begin{bmatrix} 2 & 5 & 0 \\ 5 & 3 & -1 \\ 0 & -1 & 4 \end{bmatrix}$

Skew symmetric Matrix: A square matrix is said to be skew symmetric if A^T = -A.

A = $\left[a_{ij}\right]_{m\times m}$ then $a_{ij}= -a_{ji}$ if **i** ≠ j. and a_{ij}= 0, **i** = j

For Example A=$\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$

Note : 1. Expressing A as symmetric matrix A = k (A + A')

2. Expressing A as skew symmetric matric A = k (A - A')

3. Expressing A as sum of symmetric and skew symmetric matricA = $\frac{1}{2}$(A + A') + $\frac{1}{2}$(A - A')

Determinants :

To every square matrix we can assign a number called its determinant

If A = [a_{11}], det A = | A | = a_{11}.

If A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, |A| = $a_{11}a_{22} - a_{21}a_{12}$.

Properties:

(i) The value of a determinant remains unchanged if its rows and columns are interchanged.

(ii) If any two rows/ cols. are interchanged, then sign of the determinant changes.

(iii) If two rows/ columns of a determinant are identical, value of the determinant is zero.

(iv) If all the elements of a row/ column of a determinant are multiplied by a constant k, then its value gets multiplied by k. i.e. $|kA| = k|A|$

(v) If elements of any one column (or row) are expressed as sum of two elements each, then determinant can be written as sum of two determinants.

(vi) If A & B are square matrices of same order, then $|AB| = |A|\,|B|$

Minor: Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies. Minor of an element aij is denoted by M_{ij}.

Cofactor: Cofactor of an element a_{ij}, denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is minor of a_{ij}.

Singular matrix: A square matrix 'A' of order 'n' is said to be singular, if $|A| = 0$.

Non-Singular matrix: A square matrix 'A' of order 'n' is said to be non-singular, if $|A| \neq 0$.

Adjoint of matrix: If $A = [a_{ij}]$ be a n-square matrix then transpose of cofactors of element of matrix A, is called the adjoint of A.

$$\text{adj } A = [A_{ij}]^T .$$

NOTE: $A(\text{adj}A) = (\text{adj}A)A = |A|\, I$.

Invertible Matrix: A Matrix A is said to be invertible if there exists another matrix B such that $AB = BA = I$

Inverse of a matrix: Inverse of a square matrix A exists, if A is non-singular.

$$A^{-1} = \frac{1}{|A|}\text{adj}A$$

NOTE: $|adjA| = |A|^{n-1}$, where 'n' is the order of the matrix A.

***System of Linear Equations**:

$a_1x + b_1y + c_1z = d_1$.

$a_2x + b_2y + c_2z = d_2$.

$a_3x + b_3y + c_3z = d_3$.

In Matrix form the above equations can be written as:-

$$\begin{bmatrix} a_1 & b_2 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

i.e. $AX = B$

$\Rightarrow X = A^{-1}B$ $\{ |A| \neq 0 \}$.

Criteria of Consistency.

(i) If $|A| \neq 0$, then the system of equations is said to be consistent & has a unique solution.

(ii) If $|A| = 0$ and $(\text{adj}A)B = 0$, then the system of equations is consistent and has infinitely many solutions.

(iii) If $|A| = 0$ and $(\text{adj. } A)\, B \neq 0$, then the system of equations is inconsistent and has no solution

MULTIPLE CHOICE QUESTIONS	
1.	If A is a square matrix such that $A^2 = A$, then $(I + A)^2 - 3A$ is (a) I (b) $2A$ (c) $3I$ (d) A Ans: (a), as $(I + A)^2 - 3A = I^2 + IA + AI + A^2 - 3A = I + A + A + A - 3A = I$
2.	The diagonal elements of a skew symmetric matrix are (a) all zeroes (b) are all equal to some scalar $k(\neq 0)$ (c) can be any number (d) none of these Ans: (a), as in skew symmetric matrix, $a_{ij} = -a_{ji}$ $\Rightarrow a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0$ $\Rightarrow a_{ii} = 0$, i.e. diagonal elements are zeroes.
3.	If $A = \begin{bmatrix} 5 & x \\ y & 0 \end{bmatrix}$ and $A = A'$ then (a) $x = 0, y = 5$ (b) $x = y$ (c) $x + y = 5$ (d) $x - y = 5$ Ans: (b) x = y A= $A' \Rightarrow \begin{bmatrix} 5 & x \\ y & 0 \end{bmatrix} = \begin{bmatrix} 5 & y \\ x & 0 \end{bmatrix} \Rightarrow x = y$
4.	If $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$, then write the value of x and y. (a) $\mathrm{x} = 3, \mathrm{y} = 3$ (b) $\mathrm{x} = 3, \mathrm{y} = 2$ (c) $\mathrm{x} = 2, \mathrm{y} = 2$ (d) $x = 2, y = 3$ **Ans:** (a) $\mathrm{x} = 3, \mathrm{y} = 3$ $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$ $\Rightarrow \begin{bmatrix} 2 + y & 6 \\ 1 & 2x + 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$ Comparing both matrices $\Rightarrow y = 3$ and $2x = 6$ $2 + y = 5$ and $2x + 2 = 8$ $\Rightarrow y = 3, y = 3.$ Comparing both matrices $2 + y = 5$ and $2x + 2 = 8$ $\Rightarrow y = 3$ and $2x = 6$ $\Rightarrow x = 3, y = 3.$
5.	A is a skew-symmetric matrix and a matrix B such that B'AB is defined, then B 'AB is a: (a) symmetric matrix (b) skew-symmetric matrix (c) Diagonal matrix (d) upper triangular symmetric Ans: (b) skew-symmetric matrix A is a skew-symmetric matrix ⇒ A’ = -A Consider (B’AB)’ = (AB)'(B’)’ = B’A'(B’)’ = B’A’B = B'(-A)B = -B’AB As (B’AB) = -B’AB Hence, B’AB is a skew-symmetric matrix
6.	If $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix}$, then write the value of k. (a) 17 (b) -17 (c) 13 (d) -13 Ans: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 + 4 & 1 + 10 \\ 9 + 8 & 3 + 20 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 11 \\ 17 & 23 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix} \Rightarrow k = 17$

7.	If $x\begin{bmatrix}2\\3\end{bmatrix}+y\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}10\\5\end{bmatrix}$, find the value of x. (a) 1 (b) 2 (c) 3 (d) 4 Ans: (c) 3 x$\begin{bmatrix}2\\3\end{bmatrix}+y\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}10\\5\end{bmatrix}\Rightarrow\begin{bmatrix}2x-y\\3x+y\end{bmatrix}=\begin{bmatrix}10\\5\end{bmatrix}$ By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get $2x-y=10$ ... (i) and $3x+y=5$ ... (ii) Adding Eqs. (i) and (ii), we get $5x=15\Rightarrow x=3$
8.	If $A=\begin{bmatrix}2&3\\5&-2\end{bmatrix}$ write A^{-1} in terms of A A (a)$\begin{bmatrix}2&-3\\-5&-2\end{bmatrix}$ (b)$\frac{1}{19}\begin{bmatrix}2&3\\5&-2\end{bmatrix}$ (c) $\begin{bmatrix}-2&3\\5&-2\end{bmatrix}$ (d)$\frac{1}{19}\begin{bmatrix}2&-3\\5&-2\end{bmatrix}$ Solution:-$A^{-1}=\frac{\text{Adj}(A)}{\|A\|}$ $=\frac{-1}{19}\begin{bmatrix}-2&-3\\-5&2\end{bmatrix}$ $=\frac{1}{19}\begin{bmatrix}2&3\\5&-2\end{bmatrix}$ $=\frac{1}{19}$A
9.	For what value of x, is the following matrix singular? (a)0 (b)-1 (c)2 (d)5 A=$\begin{bmatrix}3-2x&x+1\\2&4\end{bmatrix}$ Solution:- A matrix is singular if $\|A\|=0$ $(3-2x)4-(x+1)2=0$ On solving we get $x=1$
10.	If A is a nonsingular matrix of order 3 and $\|A\|=-4$, find $\|A.\,adjA\|$ (a)16 (b)32 (c)-64 (d)64 Solution:- $\|A.\,adjA\|=\|A\|\|adjA\|=\|A\|\|A\|^{2}=\|A\|^{3}=(-4)^{3}=-64$
	PRACTICE EXERCISE
1	If A is a matrix of order 3x4 and B is a matrix of order 4x3, write the order of the matrix (AB). (a)3x3 (b)4x4 (c)3x4 (d)4x3
2	If $\begin{bmatrix}2&3\\5&7\end{bmatrix}\begin{bmatrix}1&-3\\-2&4\end{bmatrix}=\begin{bmatrix}-4&6\\-9&x\end{bmatrix}$ write the value of x (a)13 (b)-4 (c) 6 (d)-9
3	If matrix A= $\begin{bmatrix}3&-3\\-3&3\end{bmatrix}$ and $A^2=\propto A$ then write the value of $\propto$ (a)9 (b) 6 (c) 18 (d)12

4	A matrix A of order 3x3 has determinant 5 what is the value of $\|3A\|$? (a)625 (b)135 (c) 45 (d) 125
5	If a matrix has 5 elements, the no of possible orders it can have (a)1 (b)2 (c)4 (d)0
ASSERTION REASON BASED QUESTIONS	
	In the following questions, a statement of Assertion(A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices (a). Both A and R are true and R is the correct explanation of A. (b). Both A and R are true and R is not the correct explanation of A. (c). A is true but R is false. (d). A is false but R is true.
1.	Assertion: $\begin{pmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{pmatrix}$is a scalar matrix. Reason: All the elements of the principal diagonal are equal, it is called a scalar matrix.
2.	Assertion: $\begin{pmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{pmatrix}$ is skew- symmetric matrix. Reason: Every square matrix A can be expressed as sum of a symmetric and skewsymmetric matrix, $\mathrm{A} = 1/2(\mathrm{A} + \mathrm{A}^{\mathrm{T}}) + 1/2(\mathrm{A} - \mathrm{A}^{\mathrm{T}})$.
3.	Assertion: If A is a square matrix such that $\mathrm{A}^2 = \mathrm{I}$, then $(\mathrm{I} + \mathrm{A})^2 - 3\mathrm{A} = \mathrm{I}$. Reason: AI=IA=A, where I is the identity matrix
4.	Assertion: If $\mathrm{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $\mathrm{A}^{\mathrm{n}} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, for all $\mathrm{n} \in \mathrm{N}$. Reason: If $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $(I + B)^n = I + nB$, for all $n \in N$.
5.	Assertion: An identity matrix is a non- singular matrix. Reason: Assertion is correct as for identity matrix I, modulus of $\mathrm{I} = 1 \neq 0$.
6.	Assertion: For two matrices A and B of order 3, $\|A\| = 3, \|B\| = -4$ then $\|2AB\| = -96$ Reason: For a matrix A of order n and a scalar $k det(kA) = k$ raised to the power n. (det A)
7.	Assertion: A inverse exists Reason: $\det A = 0$
8.	Assertion: adj A is a non- singular matrix Reason: A is non - singular matrix
9.	Assertion: $\det Q = O$ Reason: Determinant of skew symmetric matrix is 0
10.	Assertion: The value of k for which area of the triangle with vertices $(1,1), (0,2), (k, 0)$ is 3 square units Reason: We can use the determinant formula for finding area of triangle

Answers

1. (a)	2.(b)	3.(a)	4.(a)	5.(b)	6.(b)	7.(c)	8.(a)	9.(c)	10.(d)

PRACTICE EXERCISE

1\. Assertion: Given A and B are symmetric matrices of same order. If AB is symmetric ,then AB=BA

Reason: Given A and B are symmetric matrices of same order .If AB=BA then AB is symmetric

Ans (a)

2\. Assertion: Given A is matrix of order 3x3, then $(A^2)^{-1}=(A^{-1})^2$

Reason: If A is matrix of order 3x3, then the number of minors in determinant of A is 6

Ans. (c)

3\. Assertion: The adjoint of the matrices $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ is $\begin{pmatrix}4 & -2\\ -3 & 1\end{pmatrix}$

Reason: If A=$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is then adjoint A=$\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}$

Ans. (a)

4\. Assertion: $\begin{vmatrix}2 & 4\\ 5 & 1\end{vmatrix}=\begin{vmatrix}2x & 4\\ 6 & x\end{vmatrix}$then value of x is$\pm\sqrt{3}$

Reason: $\begin{vmatrix}2 & 4\\ 5 & 1\end{vmatrix}=\begin{vmatrix}2x & 4\\ 6 & x\end{vmatrix} \Mapsto 2 \times 1 - 5 \times 4 =$

$$2x \times x - 6 \times 4 \Mapsto 2 - 20 = 2x^2 - 24 \Mapsto x = \pm\sqrt{3}$$

Ans. (a)

5\. For a matrix A having $|A| = 0$ *then*

Assertion: A^{-1}does not exists

Reason: A^{-1} exists with $[A^{-1}] = 0$

Ans: (c)

VERY SHORT ANSWERS(2MARKS)

1 Construct a 2x2 matrix $A=[a_{ij}]$ whose elements are given by $a_{ij}=\frac{i}{j}$

Ans: $A = \begin{bmatrix}1 & \frac{1}{2}\\ 2 & 1\end{bmatrix}$

2\. If A is a Square matrix of order 3 and $|A| = 6$ *then* $|2A|$ ______________

Ans: $|A| = 6$ and O(A) = 3

$\therefore |2A| = 2^3|A| = 8 \times 6 = 48$

3\. If $\begin{bmatrix}5 & 3x\\ 2y & z\end{bmatrix} = \begin{bmatrix}5 & 12\\ 6 & 4\end{bmatrix}$then find the value of x, y, z

Ans. $\begin{bmatrix}5 & 3x\\ 2y & z\end{bmatrix} = \begin{bmatrix}5 & 12\\ 6 & 4\end{bmatrix}$

$\therefore 3x = 12 \Rightarrow x = 4 \text{ and } 2y = 6 \Longrightarrow y = 3 \text{ and } z = 4$

4. Ans.	If A is a square matrix of order 3 such that$\vert adjA\vert = 64, find\ \vert adjA^T\vert$. As $\vert adjA\vert = \vert A\vert^{n-1}$, $\therefore \vert A\vert^{3-1} = 64 \Rightarrow \vert A\vert = 8$ As $\vert A\vert = \vert A^T\vert \Rightarrow \vert A^T\vert = 8$
5 Ans.	Find the value of a for which $\begin{pmatrix}2a & -1\\ -8 & 3\end{pmatrix}$ $is\ singular\ matrix$ Given matrix is singular $\therefore \begin{vmatrix}2a & -1\\ -8 & 3\end{vmatrix} = 0$ $\therefore 6a - 8 = 0 \Longrightarrow a = \frac{4}{3}$
6. Ans.	For what value of K, the matrix $\begin{bmatrix}2-k & 3\\ -5 & 1\end{bmatrix}$ is not invertible? The given matrix is not invertible if $\begin{vmatrix}2-k & 3\\ -5 & 1\end{vmatrix} \neq 0 \Rightarrow -k + 17 \neq 0 \Rightarrow k \neq 17$
7. Ans.	If A is a matrix of order 2x3 and B is a matrix of order 3x5 what is the order of matrix $(AB)^T$ AB is a matrix of order 2x5 ⇒ $(AB)^T$ is a matrix of order 5x2
8. Ans.	Find the value of x :$\begin{vmatrix}x & 4\\ 2 & 2x\end{vmatrix}$ = 0 As $\begin{vmatrix}x & 4\\ 2 & 2x\end{vmatrix} = 0 \Rightarrow 2x^2 - 8 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$
9. Ans.	If A is a square matrix satisfying $A^2 = I$, then what is the A^{-1} ? As $A^2 = I \Rightarrow A^2A^{-1} = IA^{-1} \Rightarrow A(AA^{-1}) = A^{-1}$ $\Rightarrow AI = A^{-1} \Rightarrow A = A^{-1}$
10. Ans.	Evaluate $\begin{vmatrix}2cosx & -2sinx\\ sinx & cosx\end{vmatrix}$ $\begin{vmatrix}2cosx & -2sinx\\ sinx & cosx\end{vmatrix} = 2\cos^2 x + 2\sin^2 x = 2$
	PRACTICE EXERCISE
1. Ans.	Construct a 3 x 2 matrix whose elements in the ith row and jth column are given by a_{ij} =i+j $\begin{pmatrix}2 & 3\\ 3 & 4\\ 4 & 5\end{pmatrix}$
2. Ans.	If A =$\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix}$ is such that A^2=I, then the value of $1 - \alpha^2 - \beta\gamma$ 0
3. Ans.	Find a matrix X such that 3A +2B - X = 0 where A =$\begin{bmatrix}4 & 2\\ 1 & 3\end{bmatrix}$, B = $\begin{bmatrix}-2 & 1\\ 3 & 2\end{bmatrix}$. $\begin{bmatrix}8 & 8\\ 9 & 13\end{bmatrix}$
4. Ans.	Solve the matrix equation $\begin{bmatrix}x^2\\ y^2\end{bmatrix}$ - 3 $\begin{bmatrix}x\\ 2y\end{bmatrix}$ = $\begin{bmatrix}-2\\ -9\end{bmatrix}$ { (x , y) : x = 1, 2 y = 3 }
5 Ans.	If A is a square matrix$\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$such that A^2=pA, then find the value of p P=4

SHORT ANSWER TYPE PROBLEMS(3MARKS)

1. Construct a 3 x 2 matrix whose elements in the ith row and jth column are given by $a_{ij} = \frac{i+4j}{2}$.

Ans.

$$[a_{ij}]_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{9}{2} \\ 3 & 5 \\ \frac{7}{2} & \frac{11}{2} \end{bmatrix}$$

2. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$ find $3A^2 - 2B$.

Ans.

$$3A^2 = 3\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

$$= 3\begin{bmatrix} 1 & -4 \\ 12 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -12 \\ 36 & 3 \end{bmatrix} \quad \text{.......(1)}$$

$$2B = 2\begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ -2 & 14 \end{bmatrix} \quad \text{.......(2)}$$

$$(1) - (2) \rightarrow \quad 3A^2 - 2B = \begin{bmatrix} 3 & -20 \\ 38 & -11 \end{bmatrix}$$

3. Find a matrix X such that $3A - 2B + X = 0$ where $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$.

Ans.

Given $\quad 3A - 2B + X = 0$

$X = 2B - 3A$

$$= 2\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} - 3\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} -4 & 2 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} 12 & 6 \\ 3 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -16 & -4 \\ 3 & -5 \end{bmatrix}$$

4. Express A as a sum of symmetric matrix and skew symmetric matrix where $A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$.

Ans.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) A^T = \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix}$$

$$= \frac{1}{2}\left(\begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix}\right)$$

$= \frac{1}{2}\begin{bmatrix} 4 & 9 \\ 9 & 12 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = B + C$ where B is symmetric and C is skew symmetric.

5. Verify that $(AB)^T = B^T A^T$, if $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$.

Ans.

$$AB = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 4 & -2 \end{bmatrix}$$

$$\therefore (AB) = \begin{bmatrix} 17 & 0 \\ 4 & -2 \end{bmatrix}, \quad (AB)^T = \begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix} \text{......(1)}$$

$$B^T A^T = \begin{bmatrix} 1 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix} \quad \text{......(2)}$$

from (1) and (2) $\quad (AB)^T = B^T A^T$

6. Find the matrices A and B if $A+B = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}, A - B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Ans.	We have A+B=$\begin{bmatrix}7 & 0\\2 & 5\end{bmatrix}$….(1) A −B=$\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$…..(2) $Consider(1)+(2), A+B+A-B = \begin{bmatrix}7 & 0\\2 & 5\end{bmatrix}+\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix} \quad 2A = \begin{bmatrix}10 & 0\\2 & 8\end{bmatrix}$ ⇒A=$\begin{bmatrix}5 & 0\\1 & 4\end{bmatrix}$ $Consider, (1)-(2), (A+B)-(A-B) \Rightarrow$ 2B=$\begin{bmatrix}4 & 0\\2 & 2\end{bmatrix} \Rightarrow B = \begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}$.
7. Ans.	Find the matrix of A such that$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$.A=$\begin{bmatrix}4 & 1 & -2\\2 & -1 & 3\end{bmatrix}$ Left hand side matrix is of order2× 3, Thus A must be order 2 × 3, $Let\ A =$ $\begin{bmatrix}a & b & c\\d & e & f\end{bmatrix}. By\ data,$ $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}a & b & c\\d & e & f\end{bmatrix} = \begin{bmatrix}4 & 1 & -2\\2 & -1 & 3\end{bmatrix}$=$\begin{bmatrix}a+d & b+e & c+f\\d & e & f\end{bmatrix} = \begin{bmatrix}4 & 1 & -2\\2 & -1 & 3\end{bmatrix}$ Equating corresponding elements, we get d=2,e=-1,f=3,Now ,a+d=4 ⇒ $a = 2$,b+c=1⇒ $b =$ $2, c+f = -2 \Rightarrow c = -5\ thus\ the\ required\ matrix\ is\ \begin{bmatrix}2 & 2 & -5\\2 & -1 & 3\end{bmatrix}$
8. Ans.	If A=$\begin{bmatrix}4 & 2 & x\\y & 2 & -2\\3 & z & 5\end{bmatrix}$is symmetric matrix, then find x, y and z By data, A=A^T⇒ $\begin{bmatrix}4 & 2 & x\\y & 2 & -2\\3 & z & 5\end{bmatrix} = \begin{bmatrix}4 & y & 3\\2 & 2 & z\\x & -2 & 5\end{bmatrix}$ ⇒x=3,y=2,z=-2
9. Ans.	Find the integral value of x if $[x \quad 4 \quad -1]\begin{bmatrix}2 & 1 & -1\\1 & 0 & 0\\2 & 2 & 4\end{bmatrix}[x \quad 4 \quad -1]^T = O$ $[x \quad 4 \quad -1]\begin{bmatrix}2 & 1 & -1\\1 & 0 & 0\\2 & 2 & 4\end{bmatrix}[x \quad 4 \quad -1]^T$=$[x \times 2 + 4 \times 1 + (-1)2\ , x \times 1 + 4 \times 0 +$ $(-1)2, x \times (-1) + 4 \times 0 + (-1) \times 4][x \quad 4 \quad -1]^T = [2x+2 \quad x-2 \quad -x-4]\begin{bmatrix}x\\4\\-1\end{bmatrix}$=O $\Rightarrow [(2x+2)x + (x-2)4 + (-x-4)(-1)] = 0$ $\Rightarrow 2x^2 + 2x + 4x - 8 + x + 4$ =0 $\Rightarrow 2x^2 + 7x - 4 = 0$ $\Rightarrow (x+4)(2x-1) = 0$ $\Rightarrow x = -4, \frac{1}{2}\ but\ x \in I\ so\ x = -4$
10. Ans.	If A is square matrix such that A^2=A, then find the value of (I-A)^3+A I^3-A^3-3A^{2}I+3AI^2+A =I-A^2 A-3A^2+3AI+A =I-AA-3A+3A+A =I-A-3A+3A+A =I
	PRACTICE TEST/EXERCISE
1.	Construct a 3x3 matrix whose elements are given by aij=$\lvert i-j \rvert$

2.	Find the value of x if $[1 \quad x \quad 1]\begin{bmatrix}1&3&2\\2&5&1\\15&3&2\end{bmatrix}$=[0]
3.	If A=$\begin{bmatrix}1&2\\2&1\end{bmatrix}$,f(x)=$x^2$-3x+2,find f(A)
4.	If A=$\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$,verify that $(A^T)^T$=A
5.	Find the values of $2a+3b-c, if$ $\begin{bmatrix}0&-1&28\\a-8&0&3b\\-c+2&-2&0\end{bmatrix}$is a skew-symmetric.
6.	If A is a square matrix$\begin{bmatrix}2&-2\\-2&2\end{bmatrix}$such that A^2=pA,then find the value of p
7.	If A=$\begin{bmatrix}3&1\\-1&2\end{bmatrix}$and I=$\begin{bmatrix}1&0\\0&1\end{bmatrix}$,Find K so that A^2=5A +KI
8.	assume X,Y,Z,W and P are matrices of order 2×n,3×k,2×n,n×3 and p×krespectively,if n=p then find the order of 7X-5Z,
9.	If A is a square matrix such that A^2=I,Then find the value of $(A-I)^3+(A+I)^3-7A$
10.	If each dealer receive profit of Rs50000 on sale of a Hatch car, Rs 1,00,000on sale of a sedan car and Rs 1,50,000on a sale of a SUV car,then find the amount of profit received in a year 2021by each dealer.

LONG ANSWER TYPE PROBLEMS

1. Ans.	Solve the matrix equation $\begin{bmatrix}x^2\\y^2\end{bmatrix} - 3\begin{bmatrix}x\\2y\end{bmatrix} = \begin{bmatrix}-2\\-9\end{bmatrix}$ $\Rightarrow \begin{bmatrix}x^2\\y^2\end{bmatrix} - \begin{bmatrix}3x\\6y\end{bmatrix} = \begin{bmatrix}-2\\-9\end{bmatrix}$ $\Rightarrow x^2 - 3x = -2 \quad ; \quad y^2 - 6y = -9$ $\Rightarrow x^2 - 3x+2 = 0 \quad ; \quad y^2 - 6y + 9 = 0$ $\Rightarrow (x-1)(x-2) = 0 \; ; \; (y-3)^2 = 0$ $\Rightarrow x = 1,2 \quad ; \quad y = 3$ $\Rightarrow \{ (x , y) : x = 1, 2 \quad y = 3 \}$ $\Rightarrow \{(1,3),(2,3)\}$

2.

If $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 3 & -1 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system $2x - z = -4$; $x + 2y + 3z = 0$;

$2x + 2y - z + 2 = 0$.

Ans.

To find A^{-1}, $A^{-1} = \frac{1}{|A|}\text{adj } A$.

$$A^{-1} = -\frac{1}{14}\begin{bmatrix} -8 & 7 & -2 \\ -2 & 0 & -4 \\ 2 & -7 & 4 \end{bmatrix}$$

Given System,

$2x - z = -4$

$x + 2y + 3z = 0$

$2x + 2y - z = -2$

Let 'C' be its co-efficient matrix. $C = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 3 \\ 2 & 2 & -1 \end{bmatrix} = A^T$

Solution of the given system

$$\begin{aligned} X &= C^{-1} B \\ &= (A^T)^{-1}B \\ &= (A^{-1})^T B \\ &= -\frac{1}{14}\begin{bmatrix} -8 & -2 & 2 \\ 7 & 0 & -7 \\ -2 & -4 & 4 \end{bmatrix}\begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} \\ &= -\frac{1}{14}\begin{bmatrix} 28 \\ -14 \\ 0 \end{bmatrix} \end{aligned}$$

$x = -2$, $y = 1$, $z = 0$.

3.

Given $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ Verify AB = 6 I.

Ans.

Use this to solve the system

$x - y = 3$; $2x + 3y + 4z = 17$; $y + 2z = 7$.

To Verify : AB = 6 I.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \\ &= 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= 6\,I \end{aligned}$$

Verified.

Given system,

$x - y = 3$

$2x + 3y + 4z = 17$

$y + 2z = 7$

Co-efficient Matrix, $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ = A (given).

Solution of the system

$X = A^{-1} C$ where C is Constant matrix.

Since AB = 6 I ,

B = 6 A^{-1}

$A^{-1} = \frac{1}{6} B.$

$X = \frac{1}{6} B C$

$= \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

$\Rightarrow$ x = 2; y = -1; z = 4.

4. Find A^{-1} if $A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. Hence solve the system,

$2x - 3y + z = 15$; $-x + y + z = -3$; $-x + 2y + 5z = 2$.

Ans. To find A^{-1} ,

$A^{-1} = \frac{1}{|A|}\text{adj } A.$

$|A| = \begin{vmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{vmatrix}$

$= -1(-3 - 1) - 2(2 + 1) + 5(2 - 3)$

$= 4 - 6 - 5 = -7$

$A^{-1} = \frac{1}{-7}\begin{bmatrix} -4 & 3 & 17 \\ -3 & 4 & 11 \\ -1 & -1 & -1 \end{bmatrix}$

Re-arranging the system,

$-x + 2y + 5z = 2$

$2x - 3y + z = 15$

$-x + y + z = -3$

$X = A^{-1} B.$

$= \frac{1}{-7}\begin{bmatrix} -4 & 3 & 17 \\ -3 & 4 & 11 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 15 \\ -3 \end{bmatrix}$

$= \frac{1}{-7}\begin{bmatrix} -14 \\ 21 \\ -14 \end{bmatrix}$

x = 2; y = –3; z = 2

5.	Show That $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$
Ans.	$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$ $=\frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ c^2a & c^2b & c(c^2+1) \end{vmatrix}$ $=\frac{abc}{abc}\begin{vmatrix} (a^2+1) & a^2 & a^2 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$ $=\begin{vmatrix} a^2+b^2+c^2+1 & a^2+b^2+c^2+1 & a^2+b^2+c^2+1 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$ $=(a^2+b^2+c^2+1)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$ $=(a^2+b^2+c^2+1)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & -1 \\ c^2 & 0 & 1 \end{vmatrix}$ $=(a^2+b^2+c^2+1)$
6.	$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$ $=-(a+b+c)\,[(a-b)^2+(b-c)^2+(c-a)^2]$
Ans.	$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix}$ $= 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix}$ $= 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix}$ $= 2(a+b+c)[(b-c)(c-b)-(b-a)(c-a)]$ $= -(a+b+c)[2a^2+2b^2+2c^2-2ab-2bc-2ca]$ $= -(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$
7.	If x, y, z are different and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}$ = 0, then 1+xyz =0
Ans.	$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix} x-y & x^2-y^2 & x^3-y^3 \\ y-z & y^2-z^2 & y^3-z^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

$$\Rightarrow (x-y)(y-z)\begin{vmatrix} 0 & x-z & (x-z)(x+y+z) \\ 1 & y+z & y^2+yz+z^2 \\ z & z^2 & 1+z^3 \end{vmatrix}=0$$

$$\Rightarrow (x-y)(y-z)(x-z)\begin{vmatrix} 0 & 1 & (x+y+z) \\ 1 & y+z & y^2+yz+z^2 \\ z & z^2 & 1+z^3 \end{vmatrix}=0$$

$$\Rightarrow (x-y)(y-z)(x-z)(1+xyz) = 0$$

if 1+xyz = 0 (Since x,y and z are different)

8. $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$ = $(1+a^2+b^2)^3$

Ans. $\begin{vmatrix} 1+a^2+b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^3\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (1+a^2+b^2)^3$$

9. $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$ = xyz+xy+yz+zx

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = \begin{vmatrix} 1+x & -x & 0 \\ 1 & y & -y \\ 1 & 0 & z \end{vmatrix}$$

= xy + z(xy+x+y) = xy + yz + zx + xyz

PRACTICE PROBLEMS ON DETERMINANTS

1. Two schools A and B decided to award prizes to their students for three values Honesty (x), punctuality(y) and Obedience (z). School A decided to award a total of rupees 15000 for three values to 4, 3 and 2 students respectively, while school B decided to award Rs. 19000 for three values to 5,4 and 3 students respectively. If all the three prizes together amount to Rs. 5000, then represent the above situation by a matrix equation and form linear equation using matrix multiplication. Which value you prefer to be rewarded most and why?

2. Using properties of determinants, prove that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^2$$

.

3. Solve using matrix method:

(i) x+3y+4z =8, 2x+y+2z =5, 5x+y+z =7

(ii) 8x+4y+3z =18, 2x+y+z =5, x+2y+z =5

(iii) $\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 4$, $\frac{2}{x} + \frac{1}{y} + \frac{3}{2} = 0$, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$, x≠ 0, y≠ 0, z≠ 0

4. If A = $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ find A^{-1} and hence solve.$x + 2y + z = 4, -x + y + z = 0, x - 3y + z = 2$

5. If A = $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$, B = $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, find AB and use Hence to solve the following equations:$x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$

6. The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some other (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awards is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these values, namely, honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards.

PRACTICE PROBLEMS ON MATRICES

1. Express the matrix A=$\begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 6 \end{bmatrix}$as the sum of a symmetric and skew symmetric matrix.

2. If A==$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$,then show that A^3-4A^2-3A+11A=0.

3. If A=$\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, B=$\begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$Verify $(AB)^T=B^TA^T$

4.	If A=$\begin{bmatrix}0 & 2y & z\\ x & y & -z\\ x & -y & z\end{bmatrix}$is matrix which satisfies $AA^T=I_3$, then find the values of x, y, z.
5,	If A=$\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}$is matrix which satisfies $AA^T=9I$, then find the values of a and b.
6.	Express the matrix A=$\begin{bmatrix}2 & 4 & -6\\ 7 & 3 & 5\\ 1 & -2 & 4\end{bmatrix}$as the sum of symmetric and skew symmetricmatrices.
7.	Show that the matrix B^TAB is symmetric or skew symmetric accordingly when A is symmetric or skew symmetric
1. Ans.	**CASE BASED QUESTION(01)**  Read the following text and answer the following questions on the basis of the same. An industry produces only two goods x and y. The two commodities serve as intermediate input. In each other's production, 0.1 unit of x and 0.55 unit of y are needed to produce a unitof whereas 0.4 unit of x and 0.2 unit of y are needed to produce a unit of y. For final consumption to 240 units of x and 140 units of y are needed. Based on the above information answer the following 1. Find the technology matrix A 2. Evaluate the Value of I-A, where I-A is the identity matrix of the order2 3. Write the demand matrix D 4. Find the value of$(I-A)^{-1}$ 1. A=$\begin{bmatrix}0.1 & 0.4\\ 0.55 & 0.2\end{bmatrix}$ 2.$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}0.1 & 0.4\\ 0.55 & 0.2\end{bmatrix}=\begin{bmatrix}0.9 & -0.4\\ -0.55 & 0.8\end{bmatrix}$ 3.$\begin{bmatrix}240\\ 140\end{bmatrix}$ 4. $(I-A)^{-1}=\frac{1}{\lvert I-A\rvert}AdjA=\frac{1}{9.4}\begin{bmatrix}0.8 & 0.4\\ 0.55 & 0.9\end{bmatrix}$
2.	**CASE BASED QUESTION-2** Consider two families A and B. suppose there are 4 men .4women and 4 children in family A and 2men, 2womenand 2 children in family. The recommend daily amount of calories is 2400 for a

Ans.

man. 1900 for a woman.1800for a children and 45 grams of protiens for a man ,55grams for a woman and 33 grams for children.

Based on the above information, answer the following questions.

1. Represent the requirement of calories and proteins for each person as matrix form.
2. Evaluate the requirement of calories of family A.
3. If Aand B are two matrices such that AB=B and BA=A, then evaluate the value of A^2+B^2

1.$\begin{bmatrix}4&4&4\\2&2&2\end{bmatrix}\begin{bmatrix}2400&45\\1900&55\\1800&33\end{bmatrix}$

2.$\begin{bmatrix}4&4&4\\2&2&2\end{bmatrix}\begin{bmatrix}2400&45\\1900&55\\1800&33\end{bmatrix}=\begin{bmatrix}4\times2400+4\times1900+4\times1800 & 4\times45+4\times55+4\times33\\2\times2400+2\times1900+2\times1800 & 2\times45+2\times55+2\times33\end{bmatrix}=$

$\begin{bmatrix}24400&532\\12200&266\end{bmatrix}$

3

CASE BASED QUESTION (03)

If A=$(a_{ij})_{mxn}$and $B=(b_{ij})_{mxn}$are two matrices ,then A+B is of order mxn and is defined as $(a+b)_{ij}=a_{ij}+b_{ij}$where i=1,2,3…m and j=1,2,3…n

If A=$(a_{ij})_{mxn}$and $B=(b_{ij})_{nxp}$are two matrices ,then AB is of order mxp and is defined as $(AB)_{ik}=\sum_{r=1}^{n}a_{ir}b_{rk=}\,a_{i1}b_{1k}+a_{i2}b_{2k}+\ldots\ldots+a_{in}b_{nk}$

consider A=$\begin{bmatrix}2&-1\\3&4\end{bmatrix}$,B=$\begin{bmatrix}5&2\\7&4\end{bmatrix}$,C=$\begin{bmatrix}2&5\\3&8\end{bmatrix}$,D=$\begin{bmatrix}a&b\\c&d\end{bmatrix}$

Using the concept of matrices answer the following questions

1. Find the product of AB
2. Evaluate the value of a and c in matrix D such that CD-AB=O
3. Find the value of B+D

1. A=$\begin{bmatrix}2&-1\\3&4\end{bmatrix}$,B=$\begin{bmatrix}5&2\\7&4\end{bmatrix}$,

$\Rightarrow AB\begin{bmatrix}2&-1\\3&4\end{bmatrix}\begin{bmatrix}5&2\\7&4\end{bmatrix}=\begin{bmatrix}3&0\\43&22\end{bmatrix}$

2. CD=$\begin{bmatrix}2&5\\3&8\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}$=$\begin{bmatrix}2a+5c&2b+5d\\3a+8c&3b+8d\end{bmatrix}$

$\Rightarrow CD-AB=O\Rightarrow\begin{bmatrix}2a+5c&2b+5d\\3a+8c&3b+8d\end{bmatrix}-\begin{bmatrix}3&0\\43&22\end{bmatrix}=\begin{bmatrix}2a+5c-3&2b+5d\\3a+8c-43&3b+8d-22\end{bmatrix}$

$=0,2a+5c-3=0\ldots(1)3a+8c-43=0\ldots(2)2b+5d=0\ldots(3)$

$3b+8d-22=0\ldots(4)$

By soving equation 1 &2 ,3&4 a=-191,c=77,b=110,d=-44

3. B+D=$\begin{bmatrix}5&2\\7&4\end{bmatrix}$+$\begin{bmatrix}-191&110\\77&44\end{bmatrix}$=$\begin{bmatrix}186&112\\84&48\end{bmatrix}$

4.

CASE BASED QUESTION (04)

Three schools NVS, CVC and KVS decided to organize a fair for collecting money for helping the flood victims. They sold handmade fans, mats and plates from recycled material at a cost of Rs. 25, Rs. 100 and Rs. 50 each respectively. The numbers of articles sold are given as

School /Article	NVS	CVC	KVS
Handmade fans	40	25	35
Mats	50	40	50
Plates	20	30	40

1. What is the total money (in Rupees) collected by the school NVS?
   (a) 700 (b) 7000 (c) 6125 (d) 7875
2. What is the total amount of money (in Rs.) collected by schools CVC and KVS?
   (a) 14,000 (b) 15,725 (c) 21,000 (d) 13,125
3. What is the total amount of money collected by all three schools NVS,CVC and KVS?
   (a) 15,775 (b) 14,000 (c) 21,000 (d) 17,125
4. If the number of handmade fans and plates are interchanged for all the schools, then what is the total money collected by all schools?
   (a) 18,000 (b) 6,750 (c) 5,000 (d) 21,250
5. How many articles (in total) are sold by three schools?
   (a) 230 (b) 130 (c) 430 (d) 330

Answers:

1-b 2-a 3-c 4-d 5-d

PRACTICE EXERCISE

CASE STUDY :05

A manufacturer produces threes stationery products Pencil, Eraser and Sharpener which hesells in two markets. Annual sales are indicated below

Market	**Products (in numbers)**		
	Pencil	**Eraser**	**Sharpener**
A	**10,000**	**2000**	**18,000**
B	**6000**	**20,000**	**8,000**

If the unit Sale price of Pencil, Eraser and Sharpener are Rs.2.50, Rs.1.50andRs.1.00 respectively, and unit cost of the above three commodities are Rs.2.00, Rs.1.00andRs.0.50respectively, then,

Based on the above information answer the following:

1. Total revenue of market A
   a. Rs.64,000 b. Rs. 60,400 c. Rs. 46,000 d.Rs.40600
2. Total revenue of market B
   a. Rs.35,000 b.Rs. 53,000 c.Rs. 50,300 d.Rs. 30,500
3. Cost incurred in market A
   a. Rs.13,000 b.Rs.30,100 c.Rs. 10,300 d.Rs. 31,000
4. Profit in market A and B respectively area
   a. (Rs.15,000, Rs.17,000) b. (Rs.17,000, Rs.15,000)
   c. (Rs. 51,000, Rs.71,000) d.(Rs.10,000,Rs. 20,000)
5. Gross profit in both markets.
   a. Rs.23,000 b.Rs. 20,300 c.Rs. 32,000 d.Rs. 30,200

ANSWERS

1.Rs. 46,000 **2.**Rs. 53,000 **3.**RS.31,000 **4.**(Rs.15,000,Rs.17,000)
5.Rs.320

CASE BASED QUESTION 6:

Two farmers Ramakrishna and Gurucharan Singh cultivate only three varieties of rice namely Basmati, Permal and Naura. The sale (in rupees) of these varieties of rice by boththe farmers in the month of September and October are given by the following matrices AandB

September sales(inRupees)

$$A=\begin{bmatrix}10,000 & 20,000 & 30,000\\ 5000 & 30,000 & 10000\end{bmatrix}$$

October sales(inRupees)

$$B=\begin{bmatrix}5000 & 10,000 & 6,000\\ 20000 & 10000 & 10000\end{bmatrix} Ramakishan$$

$Gurucharan$

1. The total sales in September and October for each farmer in each variety can be represented as.

   a. A+B b. A-B c . A>*B* d. A<*B*

2. What is the value of A_{23}?

   a. 10000 b. 20000 c . 30000 d. 40000

3. The decrease in sales from September to October is given by______.

   a. A+B b. A-B c. A>*B* d. A<*B*

4. If Ramkishan receives 2% profit on gross sales, compute his profit for each variety sold in October.

   a.Rs. 100,Rs. 200andRs. 120 b.Rs.100, Rs. 200andRs. 130

   c.Rs. 100,Rs. 220andRs. 120 d. Rs. 110, Rs. 200andRs. 120

5. If Gurucharan receives 2% profit on gross sales, compute his profit for each variety sold in September.

   a. Rs.100, Rs. 200,Rs.120 b. Rs.1000 ,Rs.600,Rs.200

   c. Rs.400, Rs. 200,Rs.120 d. Rs.1200, Rs.200,Rs.120

ANSWERS

1.(a)A+B

2.(a)10000

3.(b)A-B

4.(a)Rs. 100,Rs. 200and Rs.120

5.(b)Rs. 1000,Rs.600,Rs.20

CHAPTER 3a: DIFFERENTIATION AND ITS APPLICATIONS

CONCEPTS, DEFINITION AND FORMULA

Differentiability at a point

Suppose $f(x)$ is a real Valued function defined on an open interval (a,b) and let $c \in (a,b)$.

Then, $f(x)$ is said to be differentiable or derivable at $x = c,\ iff \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists.

Derivative of a function:

Suppose f is a real valued function, the function defined by $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

wherever the limit exists is defined to be the derivative of f at x and is denoted by $f'(x)$ or $\frac{d(f(x))}{dx}$.

$$\boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}$$

This definition of derivative is also called the first principle of derivative.

Derivative of some standard functions:

1. $\frac{d}{dx}(x^n) = nx^{n-1}$.
2. $\frac{d}{dx}(e^x) = e^x$.
3. $\frac{d}{dx}(a^x) = a^x \log a$.
4. $\frac{d}{dx}(log\, x) = \frac{1}{x}$.
5. $\frac{d}{dx}(log_a x) = \frac{1}{x log_e a}$.
6. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

- Differentiation of a constant is zero. $ie,.$ $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(c\, f(x)) = c \frac{d}{dx}(f(x))$
- Product rule:

  $\frac{d}{dx}(f(x).g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$
- Quotient rule:$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$

DIFFERENTIATION OF A FUNCTION OF A FUNCTION

Chain Rule for Differentiation:

If $f(x)$ and $g(x)$ are differentiable functions, then fog is also differentiable and

$$(fog)'(x) = f'\big(g(x)\big)g'(x)$$

OR

$$\frac{d}{dx}\{(fog)(x)\} = \frac{d}{dg(x)}\{(fog)(x)\}\frac{d}{dx}\big(g(x)\big)$$

Differentiation of Implicit Functions:

If the variables x and y are connected by a relation of the form $f(x, y) = 0$ and it is not possible or convenient to express y as a function x in the form $y = \emptyset(x)$, then y is said to be an implicit function of x.

LOGARITHMIC DIFFERENTIATION

If the functions of the form $[f(x)]^{g(x)}$ where $f(x)$ and $g(x)$ are functions of x.

Let $y = [f(x)]^{g(x)}$. Taking logarithm of both sides, we get

$$\log y = g(x)\log\{f(x)\}$$

Then differentiate both sides w.r.t x

DIFFERENTIATION OF PARAMETRIC FUNCTIONS

Some times x and y are given as functions of a single variable. $e.g.\ x = \phi(t),\ y = \psi(t)$ are two functions of a single variable.

In such a case x and y are called parametric functions or parametric equations and t is called the parameter.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Second Order Derivative

Let $y = f(x)$. Then

$$\frac{dy}{dx} = f'(x) \ldots\ldots\ldots\ldots\ldots.(1)$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(f'(x))$$

$ie,\ \frac{d^2y}{dx^2} = f''(x)$, which is called the second derivative of $y = f(x)$.

Second Order Derivative of Parametric functions

Let $x = \phi(t)\ and\ y = \psi(t)$. Then,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

RATE OF CHANGE OF QUANTITIES

Let $y = f(x)$ be a function of x.

$\frac{dy}{dx}$=Rate of change of y with respect to x

COST AND REVENUE FUNCTION

Any manufacturing company has to deal with two types of costs, the one which varies with the cost of raw material, direct labour cost, packaging etc. is the variable cost. The variable cost is dependent on production output. As the production output increases (decreases) the variable cost will also increase (decrease). The other one is the fixed cost, fixed costs are the expenses that remain the same irrespective of production output. Whether a firm makes sales or not, it must pay its fixed costs

Cost Function: If V(x) is the variable cost of producing 'x' units and 'k' the fixed cost then, the total cost C(x) is given by C(x) = V(x) + k

Revenue Function: if R is the total revenue a company receives by selling 'x' units at price 'p' per unit produced by it then the revenue function is given by R(x) = p.x

MARGINAL COST AND MARGINAL REVENUE

Marginal cost and marginal revenue are the instantaneous rate of change of cost and revenue with respect to output i.e. rate of change of C(x) (or C), the cost function and R(x) (or R), the revenue function, with respect to production output 'x'. Therefore, the Marginal cost (MC) and the Marginal revenue (MR) are given by (Marginal cost) C'(x) =dC/dx

MR (Marginal revenue) R'(x)=dR/dx

Increasing and Decreasing Functions

Strictly Increasing Function:

A function $f(x)$ is said to be strictly increasing function on (a, b), if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$

Strictly Decreasing Function:

A function $f(x)$ is said to be strictly decreasing function on (a,b), if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a,b)$

Theorem

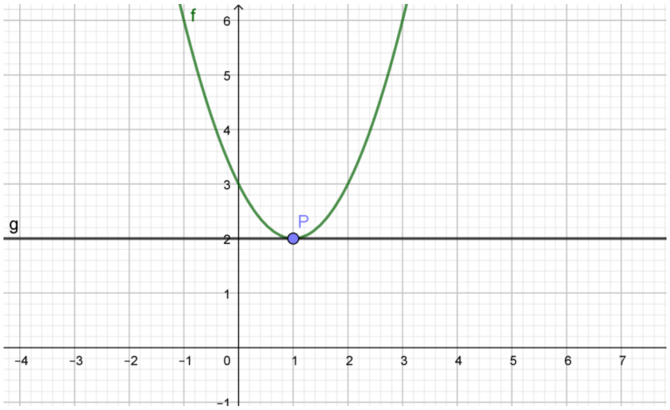
Let $f(x)$be continuous on $[a,b]$ and differentiable on (a,b). Then

- $f(x)$ is **increasing** in $[a,b]$ if $\boldsymbol{f'(x)} > 0,\ for\ each\ x \in (a,b)$
- $f(x)$ is **decreasing** in $[a,b]$ if $\boldsymbol{f'(x)} < 0,\ for\ each\ x \in (a,b)$
- $f(x)$ is a constant function in $[a,b]$ if $f'(x) = 0$, for each $x \in (a,b)$.

TANGENTS AND NORMALS

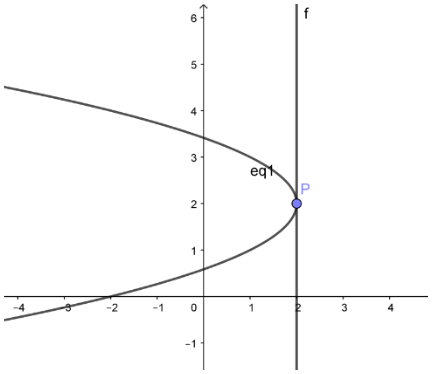
<u>Slope of the Tangent</u>:

Let $y = f(x)$ be a continuous curve, and let $P(x_1, y_1)$ be a point on it. Then $\left(\frac{dy}{dx}\right)_P$ is the slope of the tangent to the curve $y = f(x)$ at point P.



If the tangent at P is parallel to $x - axis$, then

$$slope = 0 \Rightarrow \left(\frac{dy}{dx}\right)_P = 0$$



If the tangent at P is perpendicular to $x - axis$, then

$$slope = not\ defined \Rightarrow \left(\frac{dy}{dx}\right)_P = \infty$$

$$\Rightarrow \frac{dx}{dy} = 0$$

Slope of the Normal:

The normal to a curve at $P(x_1, y_1)$is a line perpendicular to the tangent at P and passing through P.

Slope of the normal at $P = \frac{-1}{\left(\frac{dy}{dx}\right)_P}$

Equation of Tangent

The equation of the tangent at $P(x_1, y_1)$ to the curve $y = f(x)$ is given by

$$(y - y_1) = \left(\frac{dy}{dx}\right)_P (x - x_1)$$

Equation of Normal

The equation of the normal at $P(x_1, y_1)$ to the curve $y = f(x)$ is given by

$$(y - y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_P} (x - x_1)$$

MAXIMA AND MINIMA

Let *f* be a function defined on an interval I. Then

- *f* is said to have a *maximum value* in I, if there exists a point *c* in I such that $f(c) > f(x)$, for all $x \in$ I.

The number *f* (*c*) is called the maximum value of *f*in I and the point *c* is called a *point of maximum value* of *f*in I.

- (b) *f* is said to have a minimum value in I, if there exists a point *c* in I such that $f(c) < f(x)$, for all $x \in$ I.

The number *f* (*c*), in this case, is called the minimum value of *f*in I and the point *c*, in this case, is called a *point of minimum value* of *f*in I.

- (c) *f* is said to have an extreme value in I if there exists a point *c* in I such that *f* (*c*) is either a maximum value or a minimum value of *f* in I.

The number *f* (*c*), in this case, is called an *extreme value* of *f* in I and the point *c* is called an *extreme point*.

Critical point

A point c is called critical point of the function f(x) where f'(x)=0 or when f'(x) is not defined at c

First Derivative Test for Local Maxima and Minima

Let *f* be a function defined on an open interval I. Let *f* be continuous at a critical point *c* in I. Then

(i)If $f'(x)$ changes sign from positive to negative as *x* increases through c,

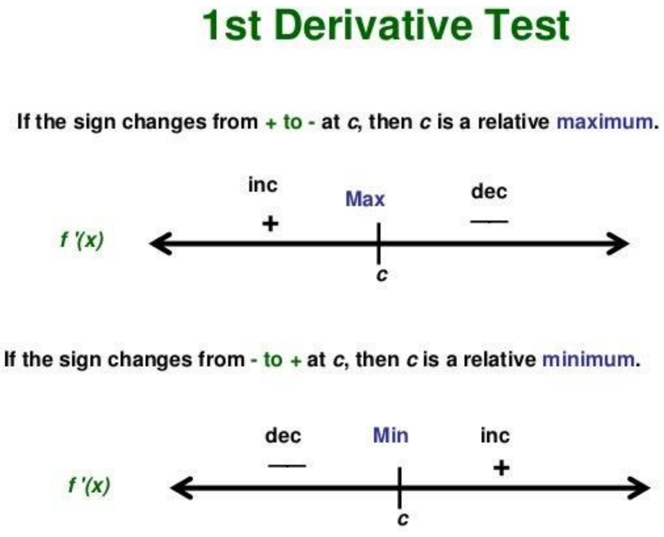
i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of *c*, and

$f'(x) < 0$ at every point sufficiently close to and to the right of *c*, then *c* is a point of *local maxima*.

(ii) If $f'(x)$ changes sign from negative to positive as *x* increases through *c*,

i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of *c*, and $f'(x) > 0$ at every point sufficiently close to and to the right of *c*, then *c* is a point of *local minima*.

(iii) If $f'(x)$ does not change sign as *x* increases through *c*, then *c* is neither a point of local maxima nor a point of local minima. In fact, such a point is called *point of inflection*



Second Derivative Test

Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $\boldsymbol{f'(c) = 0}$ and $\boldsymbol{f''(c) < 0}$

The value $f(c)$ is local maximum value of f .

(ii) $x = c$ is a point of local minima if $\boldsymbol{f'(c) = 0}$ and $\boldsymbol{f''(c) > 0}$

In this case, $f(c)$ is local minimum value of f .

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

Working rule for word problems

Step 1: Draw figure

Treat the given item as constant

Step 2: Identify the function to be maximize or minimize.

Step 3: Ensure the function is in terms of 1 variable only else convert it

Step 4: Find derivative of the function and equate to zero and solve to find the x

Step 5: Use second derivative test to confirm for maxima or minima

maxima if $\boldsymbol{f''(c) < 0}$ **and** minima if $\boldsymbol{f''(c) < 0}$

MULTIPLE CHOICE QUESTIONS

1	**1. If x – y = $\boldsymbol{\pi}$, then $\frac{dy}{dx}$ =** **a) 0** **b) 1** **c) -1** **d) 2** **Answer:** **Option B** **Solution:** 1-dy/dx=0 Then dy/dx=1

2 **If x = 4t and y = $\frac{4}{t}$, then $\frac{dy}{dx}$ =**

a. $\frac{1}{t^2}$

b. $\frac{-1}{t^2}$

c. $\frac{2}{t^2}$

d. $\frac{-2}{t^2}$

Answer:

Option B

Solution:

dx/dt=4

dy/dt=-4/t^2

dy/dx=(dy/dt)/(dx/dt)

= -1/t^2

---

3 **If e^y(x + 1) = 1, then which of the following is true:**

a. $\frac{d^2 y}{dx^2} = (\frac{dy}{dx})^2$

b. $\frac{d^2 y}{dx^2} = \frac{dy}{dx}$

c. $(\frac{d^2 y}{dx^2})^2 = (\frac{dy}{dx})^2$

d. $\frac{d^2 y}{dx^2} = \frac{dy}{dx}$

Answer:

Option A

Solution:

$e^y(x+1) = 1$

Differentiating on both sides

$e^y x + [e^y . dy/dx \ (x+1) = 0$

$$\frac{dy}{dx} = \frac{-e^y}{e^y(x+1)} = -\frac{1}{x+1}$$

Differentiating once again

$$\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} = (\frac{dy}{dx})^2$$

4 A toy manufacturing firm assesses its variable cost to be ‘x’ times the sum of 30 and ‘x’, where ‘x’ is the number of toys produced, also the cost incurred on storage is ₹ 1500. Then the marginal cost when 20 toys are produced is

a. 60 **b.70** **c. 40** **d. 15**

Answer:

Option: B

Solution:

C(x) = x (x + 30) + 1500 = x^2 + 30x + 1500

The marginal cost MC is given by

MC=dC/dx=2x+30

Marginal cost of producing 20 toys is MC(20)= 2(20)+30=70

5 Find the intervals in which the functions f(x) = x^2 – 4x + 6 is strictly increasing

(a) $(-\infty, 2) \cup (2, \infty)$ **(b)** $(2, \infty)$

(c) $(-\infty, 2)$ **(d)**$(-\infty, 2] \cup [2, \infty)$

Answer:

Option: B

Solution:

f(x)=x^2-4x+6

f ‘(x)= 2x-4

f ‘(x) =0 implies 2x-4=0, then x=2

Intervals	F ‘(x)= 2x-4	Nature
(-∞,2)	Negative	decreasing
(2, ∞)	Positive	increasing

6 The intervals in which $y = x^2e^{-x}$is increasing

(a) (−∞, ∞) **(b) (– 2, 0)**

(c) (2, ∞) **(d) (0, 2)**

Answer:

Option: D

Solution:

$y = x^2 e^{-x}$

$y' = x^2 . -e^{-x} + e^{-x} . 2x$

$y' = 0 \; implies \; x^2 . -e^{-x} + e^{-x} . 2x = 0$

$e^{-x}(-x^2 + 2x) = 0$

Implies x=0 and x=2

Intervals	$y' = x^2 . -e^{-x} + e^{-x} . 2x$	Nature
(-∞,0)	Negative	decreasing
(0,2)	Positive	increasing
(2, ∞)	Negative	decreasing

7 **The curve $y = x^{\frac{1}{5}}$ has at (0, 0)**

(a)A vertical tangent (parallel to y – axis)

(b) A horizontal tangent (parallel to x – axis)

(C)An oblique tangent

(d)No tangent

Answer:

Option:A

Solution:

$y = x^{\frac{1}{5}}$

$y' = \frac{1}{5} x^{\frac{-4}{5}} = \frac{1}{5x^{\frac{4}{5}}}$

$\frac{dy}{dx} at \; (0,0) = \infty$

Slope = implies it is vertical tangent

8 **At what point the slope of tangent to the curve $x^2 + y^2 - 2x - 3=0$ is zero ?**

(a) **(3,0), (-1, 0)**

(b) (3, 0), (1, 2)

(c) $(-\mathbf{1}, \mathbf{0}), (\mathbf{1}, \mathbf{2})$

(d) $(\mathbf{1}, \mathbf{2}), (\mathbf{1}, -\mathbf{2})$

Answer:

Option: D

Sol:

$x^2 + y^2 - 2x - 3 = 0$

$2x + 2yy' - 2 = 0$

$x + yy' - 1 = 0$

$$y' = \frac{1-x}{y}$$

$y' = 0 \; then \; x = 1 \; and \; solving \; y = 2 \; and - 2$

Therefore the points are (1,2) and (1,-2)

---

9 **The cost function of a firm is C = $3x^2 + 2x - 3$. The marginal cost , when x = 3 is:**

(a)10 **(b)25** **(c) 5** **(d) 20**

Answer:

Option: D

Solution:

C=$3x^2$+2x-3

Marginal cost , M.C =dC/dx= 6x+2

MC at x =3 is 6x3+2= 20

---

10 **The function $f(x) = x^x$ has a stationary point at**

(a) x = e **(b) x = $\frac{1}{e}$**

(c) x = 1 **(d) x = √e**

Answer: Option: b

Solution:

y=x^x

To find stationary point or critical point put dy/dx=0

Logy = x log x

	$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + logx.1$ $\frac{dy}{dx} = y(1 + logx).$ $\frac{dy}{dx} = x^x(1 + logx).$ $\frac{dy}{dx} = 0 \ implies \ x^x(1 + logx) = 0$ x^xcant be zero, 1+logx = 0 Log x= -1 Taking exponential on both sides $e^{logx} = e^{-1}$ Hence $x = \frac{1}{e}$
11	It is given that at x = 1, the function f(x) = $x^3 – 12x^2 + kx + 7$ attains maximum value, then the value of ‘k’ (a) 10 (b) 12 (c) 21 (d) 13 Answer : C
12	The maximum value of $[x(x-1)+1]^{\frac{1}{3}}, 0 \leq x \leq 1$ is (a) $(3)^{\frac{1}{3}}$ (b) $\frac{1}{2}$ (c) 1 (d) 0 Answer: C
13	The normal to the curve $x^2 = 4y$ passing through (1, 2) is (a) x + y = 3 (b) x – y = 3 (c) x + y = 1 (d) x – y = 1 Answer: A
14	Side of an equilateral triangle expands at a rate of 2cm/sec. the rate of increase of its area when each side is 10 cm is:

	(a) $10\sqrt{2}$ (b) $10\sqrt{3}$ (C)10 (d)5 Answer: B
15	If y $=\frac{4}{x^2} + \sqrt{x} - \frac{1}{\sqrt{x}}$ then y'= ? 1) $\frac{8}{x^3} + \frac{2}{\sqrt{x}} + \frac{2}{x^{3/2}}$ 2) $-\frac{8}{x^3} + \frac{1}{2\sqrt{x}} + \frac{1}{2x^{3/2}}$ 3) $-\frac{8}{x^3} + 2\sqrt{x} + 2x^{3/2}$ 4) None of these Answer: B

ASSERTION REASON BASED QUESTIONS

In the following questions, a statement of Assertion(A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices

- **a) Both A and R are true and R is the correct explanation of A.**
- **b) Both A and R are true and R is not the correct explanation of A.**
- **c) A is true but R is false.**
- **d) A is false but R is true.**

1	**Assertion (A):** The maximum profit that a company makes if profit function is given by $P(x) = 41 + 24x - 8x^2$; where 'x' is the number of units and P is the profit is 59 **Reason (R)** : The profit is maximum at x = a if $P'(a) = 0$ and $P''(a) > 0$ **Ans : Option(c)** **Assertion** : $P(x) = 41 + 24x-8x^2$ $P'(x) = 24 - 16xP'(x) = 0 \Longrightarrow 24 - 16x = 0 \Longrightarrow x = 3/2$ $P''(x) = -16 < 0 \Longrightarrow x = 3/2\ isapointofmaxima$ Max Profit = P = 41+ 24 × 3/2 − 8 × 9/4 = 41 + 36 − 18 = 59 **Assertion is true but Reason is false** , for Maximum $Profitatx = a, P'(a) = 0\ andP''(a) < 0$.
2	**Assertion (A):** If y = $log_7(x^2 + 7x + 4), then\ \frac{dy}{dx} = \frac{(2x+7)}{(x^2+7x+4),}$ **Reason(R):** $log_b a = \frac{log_e a}{log_e b}$

Ans: Option D

$y = log_7(x^2 + 7x + 4)$

$=\frac{log(x^2+7x+4)}{log7}$

$$y' = \frac{1}{log7}.\frac{1}{\log(x^2 + 7x + 4)}(2x + 7)$$

Hence Assertion is false, but Reason is correct

3 **Assertion (A):** If $y = xe^x \ then \ \frac{dy}{dx} = xe^x + e^x$

Reason(R): $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Ans: Option A

$y=xe^x$

$dy/dx= xe^x+e^x$

Both Assertion and Reason is correct

4 **Assertion(A):** The price per unit of a commodity produced by a company is given by p=30-2x and ‘x’ is the quantity demanded. The marginal revenue when 5 commodities are in demand (or produced) is 10

Reason(R): Marginal revenue when x= 5 is $\frac{d\ (30x-2x^2)}{dx}$ at x = 5 is 10

Ans: Option A

Reason: Marginal revenue = dR/dx and for when x=5 Marginal revenue =10

The revenue function R (or R(x)) is given by, R = px = (30 – 2x) x = 30x – 2x ^{2}

∴ The marginal revenue dR/dx = 30-4x

The marginal revenue of producing 5 commodities is, 30-4x5=10

5 **Assertion (A) :** $f(x) = e^x$ is an increasing function, $\forall x \in \mathcal{R}$

Reason (R) : If $f'(x) \leq 0$, then f(x) is an increasing function.

Ans: Option C

$f(x)=e^x$

$f'(x)= e^x>0$ for all x϶R, **Assertion is correct**

	Reason is wrong as f’(x)≥0, implies f is an increasing function. **Hence Assertion is correct, Reason is wrong**
6	**Assertion (A) :** $f(x) = logx$ is defined for all x∈(0, ∞). **Reason (R) :** If $f'(x) > 0$, then f(x) is strictly increasing function **Ans: Option A** F(x)= logx for all xϵ(0,∞) F’(x)=1/x , for all xϵ(0,∞) That is f(x) is increasing. That is Assertion is correct Reason is also correct. **Both Assertion and reason is correct**
7	**Assertion (A)** : For the curve $x^3 + y^3 = 6xy$, the slope of the tangent at (3, 3) is 2. **Reason (R)** :The$\left(\frac{dy}{dx}\right)_{at\ (x_{1,}y_{1,})}$gives slope of tangent of y = f(x) at (x_1, y_1). **Ans: Option D** $3x^2 + 3y^2y' = 6(xy' + y)$ $x^2 - 2y = y'(2x - y^2)$ $y' = \frac{x^2-2y}{2x-y^2},$ At(3,3) , $y' = \frac{9-6}{6-9} = \frac{3}{-3} = -1$ Hence Assertion is wrong, but reason is correct
8	**Assertion (A)**: The curves $x^3 – 3xy^2 = a$ and $3x^2y – y^3 = b$ cut each other, where ‘a’ and ‘b’ are some constants. **Reason (R)**:The given curves cut orthogonally. **Ans: Option A** $x^3 - 3xy^2 = a$ $3x^2 - 3[2xyy' + y^2] = 0$

$$y' = \frac{x^2 - y^2}{xy}$$

Slope of the tangent of the above curve at ($x_{1,}$ y_1) is

$$m_1 = y' = \frac{{x_1}^2 - {y_1}^2}{x_1 y_1}$$

Similarly for the curve

$3x^2y - y^3 = b, slope\ the\ tangent\ at$ ($x_{1,}y_1$) is

$$m_2 = y' = \frac{{y_1}^2 - {x_1}^2}{x_1 y_1}$$

Product of the slopes m_1m_2=-1

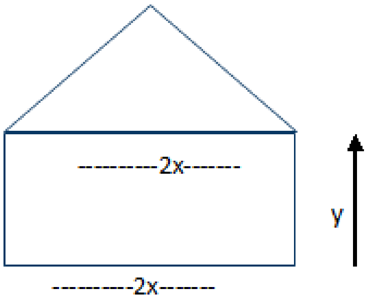
Hence the 2 curves cut orthogonally

Both Assertion and reason is correct.

9 **Assertion (A):** A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, then length $\frac{12}{6-\sqrt{3}}$m and breadth $\frac{18}{6-\sqrt{3}}$ m of the rectangle will produce the largest area of the window.

Reason (R): Critical point of a function is the value of x for which f'(x)=0

Option: B



Area of the window= Area of the rectangle + Area of the triangle

Given perimeter =14

Let the length=2x and breadth =y

Since it is an equilateral triangle all the sides are of measure 2x

2x+y+y+2x+2x+2x+2x=12 imples y=6-3x

	$Area, A = 2xy + \frac{\sqrt{3}}{4}.4x^2$ $= 2x(6-x) + \sqrt{3}x^2 = 12x - 6x^2 + \sqrt{3}x^2$ $A'(x) = 12 - 12x + \sqrt{3}.2x$ $A'(x) = 0 \; imples \; x = \frac{6}{6-\sqrt{3}}$ Hence length = $\frac{12}{6-\sqrt{3}}$ and breadth =$\frac{18-6\sqrt{3}}{6-\sqrt{3}}$ Therefore Assertion is correct Reason is also correct. But R is not the correct explanation for Assertion
10	**Assertion (A):** If $f'(x) = (x-1)^3(x-2)^8$, then f(x) has neither maximum nor minimum at x = 2. **Reason (R) :** $f'(x)$ changes sign from negative to positive at x = 2. **Ans: Option C** Given $f'(x) = (x-1)^3(x-2)^8$ F'(x) is negative to the left side of 2 and f'(x) is positive to the right side of 2 Therefore by first derivative test f is having negative Hence assumption A is wrong But reason is correct
11	**Assertion (A)**: For $f(x) = x + \frac{1}{x}, x \neq 0$, maximum and minimum values both exists. **Reason (R)**: Maximum value of f(x) is less than its minimum value. **Ans : Option(b)**
12	**Assertion (A)**: Equation of tangent at the point (2, 3) on the curve $y^2 = ax^3 + b$ is y = 4x – 5. **Reason (R):** :Value of $a = 2 \; and \; b = -7$ **Ans : Option(a)**

13	**Assertion** (A): The function $y = \log(1+x) - \frac{2x}{2+x}$ is decreasing throughout its domain. **Reason (R)**: The domain of the function $y = \log(1+x) - \frac{2x}{2+x}$ is (- 1, ∞). **Ans: Option(d)**
14	**Assertion (A):** If x = at^2 and y = 2at where ‘t’ is the parameter and ‘a’ is a constant, then $\frac{d^2y}{dx^2} = \frac{-1}{2at^3}$. **Reason(R):** $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} \div \frac{d^2x}{dt^2}$ **Ans: Option(c)**
15	**Assertion (A)**: The altitude of the cone of maximum volume that can be inscribed in a sphere of radius ‘r’ is $\frac{4r}{3}$. **Reason (R)**: The maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere **Ans: Option(b)**

VERY SHORT ANSWER TYPE QUESTIONS

1	If y = f(x) is a real function, then find derivative of the following with respect to ‘x’. f(X) = Y^2
2	Find $\frac{d^2y}{dx^2}$ of the following functions: x=at^2, y=2at.
3	Find the rate of change of volume of a sphere with respect to its surface area when the radius is 4 cm
4	Find a point on the curve y=$(x-2)^2$ at which the tangent is parallel to the chord joining the points (2,0) and (4,4).
5	Find the slope of the tangent to the curve y = $3x^4$- 4 x at x = 4.
6	Show that the function f(x) = x^7 + 8 x^5 + 1 is increasing function for all values of x.
7	Find two number whose sum is 24 and whose product is as large as possible.
8	Find the point on the curve x^2 = 8y which is nearest to the point (2,4).
9	The demand function of a toy is, x = 75 − 3p, where p is selling price of one unit and its total cost function is TC = 100 + 3x . For what value of x the profit is maximum.

10	Find the slopes of the tangent and the normal to the following curves at the indicated points. $x^2 + 3y + y^2 = 5$ at (1,1).
	SHORT ANSWER TYPE QUESTIONS
11	If $x^m . y^n = (x + y)^{(m + n)}$, then show that $\frac{dy}{dx} = \frac{x}{y}$
12	If $x^y + y^x = (a)^b$, then find $\frac{dy}{dx}$
13	Differentiate the following functions: $(2x^3 – 7)(9x^5 + 2x^2 – 3)$
14	Find the coordinates of the points on the curve $y = \frac{x}{1-x^2}$ for which $\frac{dy}{dx} = 1$
15	Find the equation of tangents to the curve $3x^2 - y^2 = 8$ which passes through the point $\left(\frac{4}{3}\ ,0)\right)$
16	Find the intervals in which the following function is strictly increasing or strictly decreasing $f(x) = 20 – 9x + 6x^2 – x^3$
17	Find the absolute maximum and absolute minimum values of a function f given by $f(x) = 2x^3 - 15x^2 + 36x + 1$ on the interval [1,5]
18	A tour operator charges Rs 136 per passenger with a discount of 40 paise for each passenger in excess of 100 . The operator requires at least 100 passengers to operate the tour . Determine the number of passengers that will maximize the amount of money the tour operator receives.
19.	A telephone company in a town has 500 subscribers on its list and collects fixed charges of Rs. 300 per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of Rs. 1, one subscriber will discontinue the service. Find what increase will bring maximum profit.
20.	The cost function for x units of a commodity is given by $C(x) = \frac{x^3}{3} + x^2 – 15x + 3$. Find marginal cost function.

VERY SHORT ANSWER TYPE QUESTIONS (UNSOLVED)

1	If y= x ^{X}, then find $\frac{dy}{dx}$
2	The total revenue received from the sale of x units of a product is given by R(x) = $3x^2 + 36x + 5$ in rupees. Find the marginal revenue when x = 5, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.
3	Show that of all rectangles with a given perimeter, the square has the largest area.
4	Find two positive numbers whose sum is 16 and whose product is as large as possible.
5.	Find critical points of the function f (x) = $x^3 – 6x^2 + 9x – 10$

SHORT ANSWER TYPE QUESTIONS (UNSOLVED)

1	A stone is dropped into a quiet lake and waves move in the form of circles at a speed of 4 cm/sec. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?
2	Determine all the points of local maxima and local minima of the following function: f(x) = (-¾)x^4 – $8x^3$ – (45/2)x^2 + 105
3	A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at a rate of 0.05 cm per second. Find the rate at which its area is increasing if the radius is 3.2 cm.
4.	A manufacturer produces x pants per week at total cost of Rs(x^2 + 78x + 2500). The price per unit is given by 8x = 600 – p, where 'p' is the price of each set. Find the maximum profit obtained, where the profit function is given by P(x) = R(x) – C(x).
5.	If price 'p' per unit of an article is p = 75x – $2x^2$ and the cost function is C(x)= 350 + 12x + $\frac{x^2}{4}$ Find the number of units and the price at which the total profit is maximum. What is the maximum profit.

Q	ANSWER KEY
	VERY SHORT ANSWER TYPE QUESTIONS
1.	In order to differentiate y^2 with respect to x , we shall have to use chain rule, as y^2 depends on y and y depends on x $\frac{d}{dx}(y^2) = \frac{dy^2}{dx}.\frac{dy}{dx} = 2y\frac{dy}{dx}$

2.	Given, x=at^2 and y=2at Then $\frac{dx}{dt}$=2at......(1) and $\frac{dy}{dt}$=2a.......(2). Now, $\frac{dy}{dx}$ ==$\frac{\frac{dx}{dt}}{\frac{dy}{dt}}$ =$\frac{1}{t}$.[Using (1) and (2)] Now, $\frac{d^2y}{dx^2}$=$\frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}$ $\frac{d^2y}{dx^2} = -\frac{1}{2at^3}$
3	$V = \frac{4}{3}\pi r^3$ S = $4\pi r^2$ $\frac{dV}{dS}$ ==$\frac{\frac{dV}{dr}}{\frac{dS}{dr}}$ $\frac{dV}{dr}$=3 X $\frac{4}{3}\pi r^2 = 4\pi r^2$ $\frac{ds}{dr}$= $8\pi r$ $\frac{dV}{dS}$=2cm^3 /cm^2
4.	y=(x−2)^2 $\frac{dy}{dx} = \frac{d(x-2)^2}{dx}$=2(x−2) ∴ slope of tangent =2x−4 Slope of line joining (2,0) and (4,4) =$\frac{4-0}{4-2} = 2$ The tangent is parallel to this line ∴ their slopes are equal 2x−4=2 ⇒2x=6 x=3 and y=(3−2)^2=1 Thus the point is (3,1)

5	The given curve is $y = 3x^4 - 4x$, thus , the slope of the tangent to the given curve at $x = 4$ is given by $\left(\frac{dy}{dx}\right)_{x=4} = [12x^3 - 4]_{x=4} = 764$
6	Here $f(x) = x^7 + 8x^5 + 1$ $f'(x) = 7x^6 + 40x^4 \geq 0$ for every x f(x) is increasing function for all values of x.
7.	Let the number be x, y Given $x + y = 24$...(1) and product $p = xy$ ....(2) From (1) we have $y = 24-x$ $\Rightarrow P = x(24-2) = 24x - x^2$ Diff wrt x $\frac{dp}{dx} = 24 - 2x$ Diff w.r.t. x $\frac{d^2p}{dx^2} = -2 < 0$ put $\frac{dp}{dx} = 0 \Rightarrow 24 - 2x = 0$ or $x = 12 \Rightarrow y = 24 - x$ or $y = 12$ by second derivative test, x=12 is a point of local maxima of f(x). Product of the number is maximum Numbers are 12 and 12.

8	Let (x, y) be the nearest to the point (2,4) Given, x^2=8y ⇒y = $x^2/8$ using distance formula, we get, $d = \sqrt{(x-2)^2 + (x-2)^2}$ Z = $d^2 = (x-2)^2 + (\frac{x^2}{8} - 4)^2$ $\frac{dZ}{dx} = \frac{x^3}{16} - 4$ $\frac{dZ}{dx} = 0$ $\frac{x^3}{16} - 4 = 0$ upon solving the above eqution, we get, x=4 , y = 2 $\frac{d^2Z}{dx^2} = \frac{x^2}{8}$ When x = 4 , $\frac{d^2Z}{dx^2} = 2 > 0, so\ the\ nearest\ point\ is\ (4,2)$
9	$TR = px = \frac{75\,x - x^2}{3}$ P = TR- TC $=\frac{75\,x - x^2}{3} - (3x + 100)$ $\frac{dp}{dx} = 22 - \frac{2}{3}\,x$ $\frac{dp}{dx} = 0$ x= 33

10

We have,

$x^2 + 3y+y^2 = 5$

Now, differentiate w.r.t x, we get

$2\,x+3\frac{dy}{dx}+2y\frac{dy}{dx}= \ 0$

$2x+(3+2y)\frac{dy}{dx} =0$

$(3+2y)\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = - \ \frac{2x}{3+2y}$

Put x=1, y=1

So,

$\frac{dy}{dx}=-3+2(1)2(1) \qquad \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{2}{5}$

Therefore,

The slope of the tangent of the curve

$=\frac{2}{5}$

The slope of the normal to the curve

$=\frac{5}{2}$

SHORT ANSWER TYPE QUESTIONS

1

Taking log on both side

$\text{Log}(\ x^{m} .\ y^{n}) = \log\,[(x + y)^{(m + n)}\ ]$

$m \log x + n \log y = (m + n) \log (\ x + y)$

differentiating wrt x

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}(1 + \frac{dy}{dx})$$

$$\frac{dy}{dx}\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \left(\frac{m+n}{x+y} - \frac{m}{x}\right)$$

$$\frac{dy}{dx}\left(\frac{n(x+y) - (m+n\,)y}{y(x+y)}\right) = \left(\frac{(m+n)x - m\,(x+y)}{x(x+y)}\right)$$

Simplifying

	$\frac{dy}{dx} = \frac{y}{x}$
2	Let $u = x^y, v = y^x$ $x^y + y^x = a^b$, where $u = x^y, v = y^x$ Differentiating both side wrt x the equation u + v = a^b $\frac{du}{dx} + \frac{dv}{dx} = 0$ --------(1) $u = x^y$ Taking log on both side Log u = y log x =>$\frac{du}{dx} = x^y \left(\frac{dy}{dx}\log x + \frac{y}{x}\right)$------------------(2) $v = y^x$ Taking log on both side log v = xlog y =>$\frac{dv}{dx} = y^x \left(\log y + \frac{x}{y}\frac{dy}{dx}\right)$------------------(3) using (2) and (3) in (1) $x^y \left(\frac{dy}{dx}\log x + \frac{y}{x}\right) + y^x \left(\log y + \frac{x}{y}\frac{dy}{dx}\right)$ = 0 $\frac{dy}{dx} = \frac{x^{y-1}\, y + \log y . y^x}{x^y \log x + x\, y^{x-1}}$
3	$y = (2x^3 - 7)(9x^5 + 2x^2 - 3)$,differentiating wrt x $\frac{dy}{dx} = (2x^3 - 7)\frac{d}{dx}(9x^5 + 2x^2 - 3) + (9x^5 + 2x^2 - 3)\frac{d}{dx}(2x^3 - 7)$ $\frac{dy}{dx} = (2x^3 - 7)(45x^4 + 4x) + (9x^5 + 2x^2 - 3)(6x)$ $= 144x^7 - 295x^4 - 18x^2 - 28$

4	$y = \frac{x}{1-x^2}$ =>$\frac{dy}{d\,x} = \frac{(1-x^2).1-x.(-2x)}{(1-x^2)^2}$ =>$\frac{dy}{d\,x} = \frac{1+x^2}{(1-x^2)^2}$ Now $\frac{dy}{d\,x} = 1$ =>$\frac{1+x^2}{(1-x^2)^2}$ =1 ⇨ x^4 -3x^2 = 0 = > x= 0, $\pm\sqrt{3}$ ⇨ when x=0 , y = 0 ⇨ when x = + $\sqrt{3}$; y = - $\frac{1}{\sqrt{3}}$ ⇨ when x = -$\sqrt{3}$; y = $\frac{1}{\sqrt{3}}$ ⇨ hence the points are (0,0),($\sqrt{3}$;- $\frac{1}{\sqrt{3}}$) ,),($-\sqrt{3}$; $\frac{1}{\sqrt{3}}$)
5	$3x^2 - y^2 = 8$ Slope of tangent at (x, y) (point of contact) will be $\frac{3x}{y}$ …..(1)(by differentiating the curve equation) Tangent also passes through ($\frac{4}{3}$, 0) so slope will be $\frac{y-0}{x-\frac{4}{3}}$ … …...(2) Equating (1) &(2) We get x=2, y=2,−2 So equation of tangents will be y = 3x − 4, y = − 3 x + 4
6	f ‘(x) = 0 => -9+ 12 x -3 x^2 -3(x^2 -4x + 9) = 0 -(3)(x -1)(x-3) = 0 X = 1,-3 The interval (- ∞, 1)(1, 3),(3, ∞.) So the function is strictly increaing in (1, 3) and decreasing in (- ∞, 1) and (3, ∞.)
7	f(x) = 2 x^3 - 15 x^2 + 36 x + 1 f’(x) = 6x^2 – 30x + 36 f’(x) = 6(x^2 – 5x + 6) = 6 (x – 2)(x – 3) Note that f’(x) = 0 Gives, x = 2 and x = 3 We shall now evaluate the value of f at these points and at the end points of the interval [1, 5], i.e. at x = 1, 2, 3 and 5

	At x = 1, f(1) = 2(1) – 15(1) + 36(1) + 1 = 24 At x = 2, f(2) = 2(8) – 15(4) + 36(2) + 1 = 29 At x = 3, f(3) = 2(27)– 15(3)2+ 36(3) + 1 = 28 At x = 5, f(5) = 2(125)– 15(25)+ 36(5) + 1 = 56 Thus, We conclude that the absolute maximum value of f on [1, 5] is 56, occurring at x = 5, and absolute minimum value of f on [1, 5] is 24 which occurs at x = 1
8	Tour operator charges $= 136 - \frac{40}{100}(x - 100), for\ x \geq 100$ $= 136 - \frac{4x}{10} + \frac{4}{10} \times 100$ $=176 - \frac{2x}{5}$ =amount of money, A = (number of passengers) x (tour operator charges) $A = x\left(176 - \frac{2x}{5}\right)$ $A = \left(176x - \frac{2x^2}{5}\right)$ $\frac{dA}{dx} = 176 - \frac{4x}{5}$ When $\frac{dA}{dx} = 0\ we\ get$ $176 - \frac{4x}{5} = 0$ X = 220 $\frac{d^2A}{dx^2} = -\frac{4}{5}\ , negative$ The amount of money is maximum when the number of passengers is 220.
9	Consider the company increases the annual subsrciption by x. So, x subscribers will discontinue the service. ∴ Total revenue of company after the increment is given by $R(x)=(500-x)(300+x)=15\times10000+500x-300x-x^2$

	$= -x^2+200x+150000$ On differentiating both sides w.r.t. x, we get $R'(x)=-2x+200$ Now,$R'(x)=0 \Rightarrow 2x=200 \Rightarrow x=10$ $\therefore R''(x)=-2<0$ So, R (x) is maximum when x = 100.
10	$C(x) = \frac{x^3}{3} + x^2 - 15x + 3.$ Marginal cost = $\frac{d}{dx}(\frac{x^3}{3} + x^2 - 15x + 3)$ $=(\frac{3x^2}{3} + x2 - 15)$ $=(3x^2 + x2 - 15$
	ANSWER (Very Short Answer Type Questions)2 mark(unsolved)
1	$dy/dx=x^x(1+\log x)$
2	126
3	Show that
4	8 and 8
5	1,3
	ANSWER (Short Answer Type Questions)3 mark (unsolved)
1	$dA/dt = 80\pi\ cm^2/sec$
2	f ″(0) = – 45 < 0. Hence, x = 0 is point of local maxima f ″(–3) = 18 > 0. Hence, x = –3 is point of local minima f ″(–5) = –30 < 0. Hence, x = –5 is point of local maxima.
3	$\frac{dy}{dx} = 0.320\ \pi cm^2/sec$
4	Rs.5069
5	X= 14

CASE BASED QUESTIONS:

1

A company is selling a certain product. The demand function for the product is linear. The company can sell 2000 units when the price is ₹ 8 per unit and it can sell 3000 units when the price is ₹ 4 per unit. Based on the above information , answer the following questions

(a) Find the demand function and total revenue function

OR

Find the value of x for which the total revenue will be maximum

(b) Find the range of values of x for which revenue is increasing

(c) Find the maximum revenue

ANS:

Let the demand function be $p = ax + b$ (given it is linear) where p is the price per unit and x is the demand.

(a) Given that when $x = 2000\ \ p = 8\ \ and$ when $x = 3000\ \ p = 4$

$8 = 2000a + b$ .... (1)

$4 = 3000a + b$ ....(2)

Solving (1) and (2) $a = -\frac{1}{250}$ and $b = 16$

Hence the demand function is $p = -\frac{1}{250}x + 16$

Total revenue function $R(x) = px = -\frac{1}{250}x^2 + 16x$

OR

$$R'(x) = -\frac{1}{250} \times 2x + 16 = -\frac{x}{125} + 16$$

$$R'(x) = 0 \Rightarrow -\frac{x}{125} + 16 = 0$$

$$\Rightarrow x = 16 \times 125 = 2000$$

$$R''(x) = -\frac{1}{125}$$

$R''(2000) = -\frac{1}{125} < 0 \Rightarrow R(x)$ is maximum when $x = 2000$

Hence revenue is maximum at $x = 2000$

(b) For revenue to be increasing, $R'(x) \geq 0 \Rightarrow -\frac{x}{125} + 16 \geq 0$

$$\Rightarrow x \leq 2000$$

(c) Revenue is maximum at $x = 2000$. So maximum revenue = $R(2000) = -\frac{1}{250} \times (2000)^2 + 16 \times 2000$

$$= -16000 + 32000$$

$= ₹16000$.

2 **2) Read the following passage and answer the questions given below.**

The relation between the height of the plant ($y\ cm$) with respect to exposure to sunlight is governed by the equation $y = 4x - \frac{1}{2}x^2$, where x is the number of days exposed to light.

Based on the above information, answer the following questions.

(a) Find the rate of growth of the plant with respect to number of days exposed to sunlight

(b) What will be the height of the plant after 2 days ?

(c) What is the maximum number of days it will take for the plant to grow to the maximum height?

What is the maximum height of the plant?

OR

If the height of the plant is $\frac{7}{2}$ cm, find the number of days it has been exposed to the sunlight.

ANS:

(a) $y = 4x - \frac{1}{2}x^2$ where x is the number of days and h is the height of the plant .

Rate of growth = $\frac{dy}{dx} = 4 - \frac{1}{2} \times 2x = 4 - x$

(b) Height after 2 days $4 \times 2 - \frac{1}{2}(2^2) = 6\ cm$

(c) $\frac{dy}{dx} = 0 \Rightarrow 4 - x = 0 \Rightarrow x = 4$

$\frac{d^2y}{dx^2} = -1$. So $\left[\frac{d^2y}{dx^2}\right]_{x=4} = -1 < 0$

Hence height of the plant is maximum at $x = 4$

Maximum height = $4 \times 4 - \frac{1}{2} \times (4)^2 = 8\ cm$

OR

Given $y = \frac{7}{2}$ then $\frac{7}{2} = 4x - \frac{1}{2}x^2 \Rightarrow x^2 - 8x - 7 = 0$

$\Rightarrow (x - 1)(x - 7) = 0$

$\Rightarrow x = 1, 7$. But the maximum height obtained at $x = 4$.

So $x = 7$ is not possible. Hence $x = 1$

3 **3) The shape of a toy is given as $f(x) = 6(2x^4 - x^2)$. To make the toy beautiful 2 sticks which are**

perpendicular to each other were placed at a point (2,3), above the toy.

Based on the above information answer the following questions.

(a) Find the values of abscissa at critical points.

OR

Find the interval where $f(x)$ is increasing?

(b) Find the slope of the normal based on the position of the stick.

(c) What will be the equation of the tangent at the critical point if it passes through (2,3).

ANS:

$$f(x) = 6(2x^4 - x^2)$$

$$f'(x) = 6(2 \times 4x^3 - 2x) = 12x(4x^2 - 1) = 48x\left(x^2 - \tfrac{1}{4}\right) = 48x\left(x + \tfrac{1}{2}\right)\left(x - \tfrac{1}{2}\right)$$

$$f'(x) = 0 \Longrightarrow 48x\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right) = 0$$

$$\Longrightarrow the\ critical\ points\ are\ \ x = 0, x = -\frac{1}{2}\ , x = \frac{1}{2}$$

OR

	$\left(-\infty, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, 0\right)$	$\left(0, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \infty\right)$
48	+	+	+	+
x	-	-	+	+
$\left(x + \frac{1}{2}\right)$	-	+	+	+
$x - \frac{1}{2}$	-	-	-	+
$f'(x)$	-	+	-	+

$f(x)$ is increasing in $\left[-\frac{1}{2}, 0\right] \cup \left[\frac{1}{2}, \infty\right)$

(b) Slope of the tangent at the position of the stick (2,3) = $f'(2) = 6(2 \times 2^4 - 2^2) = 360$

Hence slope of the normal = $-\frac{1}{360}$

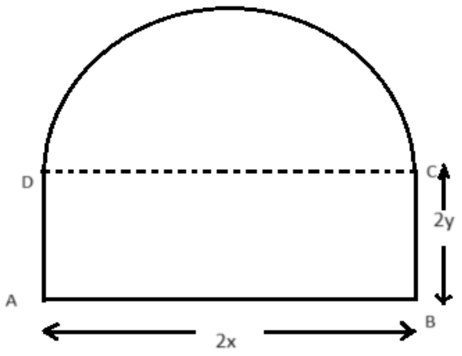
4

4) An architect designs a building for a small company. The design of window on the ground floor is

proposed to be different than other floors. The window is in the shape of a rectangle which is

surmounted by a semi-circular opening . The window is having a perimeter of 10 meter as

shown in the figure.



Based on the above information answer the following

(i) Find the relation between the variables, if $2x$ and $2y$ represents the length and breadth of the rectangular portion of the window.

(ii) Find the combined area(A) of the rectangular region and semi-circular region of the window expressed as a function of x.

(iii) Find the length of the rectangular portion of the window should be, if the owner of this small company is interested in maximizing the area of the whole window so that maximum light input is possible.

OR

Find the maximum area of the whole window.

ANS:

(i) $(2+\pi)x+4y=10$

(ii) Area = $10x-\frac{(4+\pi)x^2}{2}$

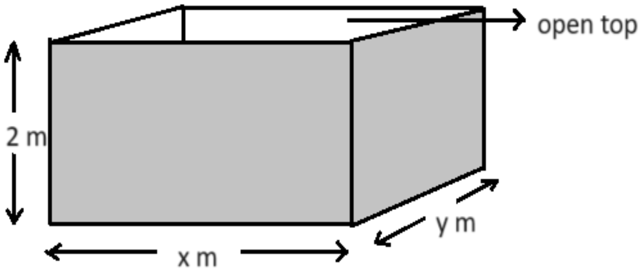
(iii) $x=\frac{10}{4+\pi}$. ∴ length of the rectangular portion = $2x=\frac{20}{4+\pi}$

OR

Maximum area = $\frac{50}{4+\pi}$

5

5) A factory owner wants to construct a tank with rectangular base and rectangular sides, open at the top, so that its depth is 2m and capacity is 8 m^3. The building of the tank costs ₹ 280 per square meter for the base and₹ 180 per square meter for the sides.



Based on the above information , answer the following questions

(a) If the length and the breadth of the rectangular base of the tank are x meters and y meterrespectively, then find a relation between x and .

(b) If C (in ₹) is the cost of construction of the tank, then find C as a function of x .

(c) Find the value of x for which the cost of construction of the tank is least.

OR

Find the least cost of construction of the tank

ANS:

(a) $xy=4$

(b) Total cost = $1120+720\left(x+\frac{4}{x}\right)$

(c) $x=2$

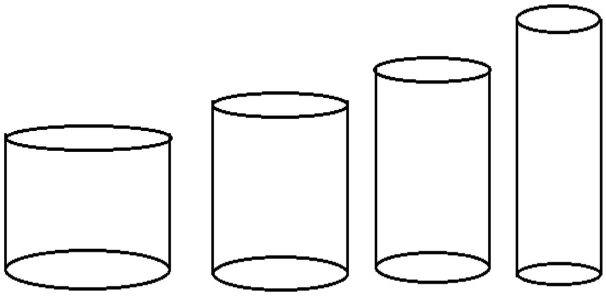
OR

Minimum cost = ₹4000

6 **A company is planning to launch a new product and decides to pack the new product in closed right**

circular cylindrical cans of volume $432\pi\ cm^3$. The cans are to be made from tin sheet. The company

tried different options.



Based on the above information, answer the following questions:

(a) If r cm is the radius of the base of the cylinder and h cm is height, then find a relation between r and h.

(b) If $S\ cm^2$ is the surface area of the closed cylindrical can, then find S in terms of r.

(c) Find the minimum surface area of cylindrical can

LONG ANSWER QUESTION

1 **If $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$, find $\frac{d^2y}{dx^2}$ at $t = 2$**

ANS:

$x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$

$$\frac{dx}{dt} = \frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2} = \frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2} = \frac{-4t}{(1+t^2)^2}$$

$$\frac{dy}{dt}=\frac{(1+t^2)2-2t(2t)}{(1+t^2)^2}=\frac{2+2t^2-4t^2}{(1+t^2)^2}=\frac{2-2t^2}{(1+t^2)^2}=\frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{2(1-t^2)}{(1+t^2)^2}}{\frac{-4t}{(1+t^2)^2}}=\frac{2(1-t^2)}{-4t}=-\frac{1}{2}(\frac{1}{t}-t)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(-\frac{1}{2}\left(\frac{1}{t}-t\right)\right)=\frac{d}{dt}\left(-\frac{1}{2}\left(\frac{1}{t}-t\right)\right)\frac{dt}{dx}=-\frac{1}{2}\left(-\frac{1}{t^2}-1\right)\frac{(1+t^2)^2}{-4t}$$

$$=-\frac{1}{8t^3}(1+t^2)^3$$

$$\frac{d^2y}{dx^2}_{t=2}=-\frac{(1+2^2)^3}{8\times 2^3}=-\frac{125}{64}$$

2 **Find the equation of the normal to the curve $x^2+2y^2-4x-6y+8=0$ at the point whoseabscissa is 2.**

ANS:

Given curve $x^2+2y^2-4x-6y+8=0$

when $x=2$

$$2^2+2y^2-4\times 2-6y+8=0$$

$$2y^2-6y+4=0$$

$$y^2-3y+2=0\Longrightarrow (y-1)(y-2)=0$$

$$y=1,\qquad y=2$$

Hence the points of contact are $(2,1)$ and $(2,2)$

$$\frac{d}{dx}(x^2+2y^2-4x-6y+8)=0$$

$$2x+4y\frac{dy}{dx}-4-6\frac{dy}{dx}=0\Longrightarrow x+2y\frac{dy}{dx}-2-3\frac{dy}{dx}=0$$

$$\frac{dy}{dx}(2y-3)=2-x$$

$$\frac{dy}{dx}=\frac{2-x}{2y-3}$$

At (2,1)

Slope of the tangent =$[\frac{dy}{dx}]_{(2,1)} = \frac{2-2}{2\times1-3} = 0$

Hence tangent at (2,1) is parallel to X-axis so that normal is perpendicular to X-axis or parallel to Y-axis passing through the point (2,1). Hence the equation of the normal at (2,1) is $x = 2$

At (2,2)

Slope of the tangent =$[\frac{dy}{dx}]_{(2,2)} = \frac{2-2}{2\times2-3} = 0$

Hence the equation of the normal at (2,2) is $x = 2$.

So the curve has same normal at (2,1) and (2,2)

3 **Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1,2). Also findthe equation of the corresponding tangent.**

ANS:

$$x^2 = 4y \Longrightarrow y = \frac{x^2}{4}$$

$$\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$$

Slope of the tangent at (1,2) = $[\frac{dy}{dx}]_{(1,2)} = \frac{1}{2}$

And slope of the normal at (1,2) = -2

Equation of tangent at (1,2) is $y - 2 = \frac{1}{2}(x - 1)$

$$2y - 4 = x - 1$$

$$x - 4y + 3 = 0$$

Equation of normal at (1,2) is $y - 2 = -2(x - 1)$

$$y - 2 = -2x + 2$$

$$2x + y - 4 = 0$$

4

	Find the intervals in which the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is strictly increasing or decreasing. Also find the points on which the tangents are parallel to x-axis. **ANS:** Strictly increasing in $(-\infty, 2] \cup [3, \infty)$ Strictly decreasing in $[2,3]$ The points at which the tangent parallel to the x-axis are (2,29) and (3,28)
5	**A firm has the cost function $C = \frac{x^3}{3} - 7x^2 + 111x + 50$ and demand function $x = 100 - p$** **(i) Write the total revenue function in terms of x** **(ii) Formulate the total profit function P in terms of x** **(iii) Find the profit maximizing level of output x.** **What is the maximum profit ?** **ANS:** (i) Revenue function = $100x - x^2$ (ii) Profit function = $-\frac{x^3}{3} + 6x^2 - 111x - 50$ (iii) $x = 11$ (iv) Maximum Profit = 111.3

UNIT-3B: INTEGRATION AND ITS APPLICATIONS

Definitions & Formulae:

- If $\frac{\boldsymbol{dF(x)}}{\boldsymbol{dx}} = \boldsymbol{f(x)}$, then we say that the integral or primitive or antiderivative of $f(x)$ with respect to x is $F(x)$ and, symbolically we write $\int \boldsymbol{f(x)dx} = \boldsymbol{F(x)}$.
- The process of integration and differentiation are inverses of each other.

	Differentiation			
Integral Primitive Antiderivative	$\boldsymbol{x^2}$		$\boldsymbol{2x}$	Derivative or Differential Coefficient
	$\boldsymbol{x^2+1}$			
	$\boldsymbol{x^2-1}$			
	$\boldsymbol{x^2+3}$			
	$\boldsymbol{x^2-4}$			
	$\boldsymbol{x^2+c}$			
	Integration			

- **Formulae**

$\int dx = x + c$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
$\int e^{ax} dx = \frac{e^{ax}}{a} + c$	$\int a^x dx = \frac{a^x}{loga} + c$
$\int k\ dx = kx, where\ k\ is\ a\ constant$	$\int \frac{1}{x} dx = \log\|x\| + c$
$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left\|x+\sqrt{x^2+a^2}\right\| + c$	$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left\|x+\sqrt{x^2-a^2}\right\| + c$
$\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left\|x+\sqrt{x^2+a^2}\right\| + c$	
$\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left\|x+\sqrt{x^2-a^2}\right\| + c$	

$\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$	
$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left\lvert\frac{x-a}{x+a}\right\rvert + c$	$\boldsymbol{\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert + c}$
$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$	

- Properties of Indefinite Integral:

$\frac{d}{dx}(\int f(x)) = f(x)$	$\int \frac{d}{dx}\big(f(x)\big) = f(x) + c$
$\int k.f(x)dx = k\int f(x)dx$	$\int [f(x) + g(x)]dx \ = \int f(x)dx + \int g(x)dx$
$\int [af(x) + bg(x)]dx = a\int f(x)dx + b\int g(x)dx$	
$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\left(\int g(x)dx\right)dx$	
$\int e^x[f(x) + f'(x)]dx = e^x.f(x) + c$	
$\int f(x)dx = F(x) + c\ , then\ \int_a^b f(x)dx = F(b) - F(a)$	

Properties of definite integrals

$\int\limits_a^b f(x)dx = \int\limits_a^b f(t)dt$	$\int\limits_a^b f(x)dx = -\int\limits_b^a f(x)dx$
$\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx$	$\int\limits_a^b f(x)dx = \int\limits_a^b f(a+b-x)dx$
$\int\limits_0^a f(x)dx = \int\limits_0^a f(a-x)dx$	$\int\limits_0^{2a} f(x)dx = \int\limits_0^a f(x)dx + \int\limits_0^a f(2a-x)dx$
$\int\limits_0^{2a} f(x)dx = \begin{cases} 2\int\limits_0^a f(x)dx & if \quad f(2a-x) = f(x) \\ \quad 0 & if \quad f(2a-x) = -f(x) \end{cases}$	

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_{0}^{a} f(x)dx & if \quad f(-x) = f(x) \quad or\ f(x)\ is\ even\ function \\ 0 & if \quad f(-x) = -f(x) \quad or\ f(x)\ is\ a\ odd\ function \end{cases}$$

Consumers’ Surplus And Producers’ Surplus Formula

$$Consumers\ surplus = \int_{0}^{x_o} pdx - p_o x_o$$

$$Producers\ surplus = p_o x_o - \int_{0}^{x_o} pdx$$

MULTIPLE CHOICE QUESTIONS:

1.	$\int x\sqrt{x+2}dx$ (A)$\frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{2}{3}(x+2)^{\frac{3}{2}} + c$ (B)$\frac{5}{2}(x+2)^{\frac{5}{2}} + \frac{3}{2}(x+2)^{\frac{3}{2}} + c$ (C)$\frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + c$ (D)$\frac{2}{5}(x+2)^{\frac{5}{2}} + \frac{4}{3}(x+2)^{\frac{3}{2}} + c$
2.	$\int \frac{dx}{\sqrt{x}+x} is\ equal\ to$ (A)$log\lvert\sqrt{x}\rvert + c$ (B)$log\lvert 1+\sqrt{x}\rvert + c$ (C)$2.\,log\lvert 1+\sqrt{x}\rvert + c$ (D)$4.\,log\lvert 1+\sqrt{x}\rvert + c$
3.	If $\frac{d}{dx}[f(x)] = ax + b\ and\ f(0) = 0,$ then f(x) equal to (A) a+b (B)$\frac{a}{2}x^2 + bx$ (C)$\frac{a}{2}x^2 + bx + c$ (D) b
4.	$\int \frac{1}{x(1+logx)}dx\ \ is\ equal\ to$ (A)$log\lvert 1+logx\rvert + c$ (B)$\frac{1}{1+logx}$ (C)$log\lvert x(1+logx)\rvert + c$ (D) None of the above
5.	$\frac{d}{dx}\big(f(x)\big) = logx, then\ f(x) equals:$ (A)$-\frac{1}{x} + c$ (B)$x(logx-1) + c$ (C)$x(logx + x) + c$ (D)$\frac{1}{x} + c$
6.	$\int x^4 logx\ dx\ \ is\ equal\ to$

(A)$\frac{x^5}{5}(logx - 5) + c$ (B)$\frac{x^5}{5}(logx + 5) + c$

(C)$\frac{x^5}{5}\left(logx - \frac{1}{5}\right) + c$ (D)$\frac{x^5}{5}\left(logx + \frac{1}{5}\right) + c$

7. $\int (x - 1)e^{-x}dx$is equal to

(A)$(x - 2)e^{-x} + c$ (B)$xe^{-x} + c$

(C)$-xe^{-x} + c$ (D)$(x + 1)e^{-x} + c$

8. $\int \frac{x - 5}{(x - 3)^3} e^x \, dx \;\; is \; equal \; to$

(A)$\frac{e^x}{x-3} + c$ (B)$\frac{e^x}{(x-3)^2} + c$

(C)$\frac{e^x}{(x-3)^3} + c$ (D) None of the above

9. $\int_1^3 (x^2 + 1)dx \; is \; equal \; to$

(A)$\frac{16}{3}$ (B)$\frac{22}{3}$ (C)$\frac{32}{3}$ (D)$\frac{34}{3}$

10. $If \int_0^a 3x^2dx = 8, then \; the \; value \; of \; a \; is \; equal \; to$:

(A) 2 (B) 4 (C) 8 (D) 10

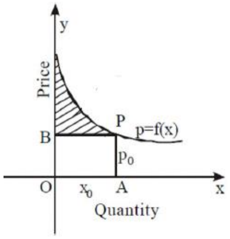
11. If $\frac{4x-10}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$, $the \; values \; of \; A \; and \; B \; respectively \; are$

(A) $2, -6$ (B) $2,6$ (C) $-2,6$ (D) $-2, -6$

12. Suppose that demand is given by the equation x_d=500 – 50P, where xd is quantity demanded, and P is the price of the good. Supply is described by the equation $x_s = 50$ + 25P where xs is quantity supplied. What is the equilibrium price

(A) 100 (B) 200 (C) 250 (D) 300

13. The shaded region in the given figure represents



(A) Equilibrium Price

(B) Producers’ Surplus

(C) Consumers’ Surplus

(D) None of the above.

Hints and Answers

Q1.	Assumption: t=x+2
Q2.	$Rewrite \int \frac{dx}{\sqrt{x} + x} \quad as \quad \int \frac{dx}{\sqrt{x}(1 + \sqrt{x})}$

	Assume $1+\sqrt{x}=t$
Q3	$\frac{d}{dx}[f(x)] = ax+b \Rightarrow f(x) = \int (ax+b)dx = \frac{ax^2}{2}+bx+c$ $f(0)=0 \ \Rightarrow c=0. f(x) = \frac{ax^2}{2}+bx$
Q4.	Assumption $1+logx=t$
Q5.	$\frac{d}{dx}\left(f(x)\right) = logx, then\ f(x) = \int logx.dx$ Apply Integral by parts by taking as $f(x) = logx\ and\ g(x) = 1$ $\int logxdx = logx.x - \int \frac{1}{\cancel{x}}.\cancel{x}\,dx = x.logx - x + c = x(logx-1)+c$
Q6.	$\int \quad x^4 logx\ dx = logx.\frac{x^5}{5} - \int \frac{1}{x}\frac{x^5}{5}dx = \frac{x^5}{5}logx - \frac{x^5}{25} + c = \frac{x^5}{5}\left(logx - \frac{1}{5}\right) + c$
Q7.	$\int \quad (x-1)e^{-x}dx = \int e^t(-t-1)dt$ Let $-x=t \ \Rightarrow x=-t \ \Rightarrow dx=-dt$ $e^t(-t)+c = e^{-x}(-x)+c$
Q8.	$\int \frac{x-5}{(x-3)^3}e^x\,dx = \int e^x\left[\frac{x-3}{(x-3)^3} - \frac{2}{(x-3)^3}\right]dx$ $\int e^x\left[\frac{1}{(x-3)^2} - \frac{2}{(x-3)^3}\right]dx; Let\ f(x) = \frac{1}{(x-3)^2}\ \ the\ f'(x) = -\frac{2}{(x-3)^3}$ $\int e^x\left[\frac{1}{(x-3)^2} - \frac{2}{(x-3)^3}\right]dx = e^x\frac{1}{(x-3)^2}+c$
Q9.	$\int_1^3 (x^2+1)dx = \left[\frac{x^3}{3}+x\right]_1^3 = (9+3) - \left(\frac{1}{3}+1\right) = 12 - \frac{4}{3} = \frac{32}{3}$
Q10.	$\int_0^a 3x^2dx = 8 \ \Rightarrow \ [x^3]_0^a = 8 \ \Rightarrow a^3 = 8 \Rightarrow a = 2$
Q11.	For A: $\left[\frac{4x-10}{x-4}\right] at\ x=3 \ \Rightarrow A = \frac{12-10}{3-4} = -2$ For B: $\left[\frac{4x-10}{x-3}\right] at\ x=4 \ \Rightarrow B = \frac{16-10}{4-3} = 6$
Q12.	At equilibrium price demand= supply. 500-50p=50+25p ➔75p=450 ➔ p=6 Then $x_d = x_s = 200$

Answers:

1.C	2.C	3.B	4.A	5.B
6.C	7.C	8.B	9.C	10.A
11.C	12.B	13.C		

Practice Questions:

1.	$\int x^2e^{x3}\, dx$ *is equal to* (A)$\frac{1}{3}e^{x3}+c$ (B)$\frac{1}{3}e^{x4}+c$ (C)$\frac{1}{2}e^{x3}+c$ (D)$\frac{1}{2}e^{x2}+c$
2.	$\int \frac{x-1}{\sqrt{x+4}}dx$ is equal to (A)$\frac{2}{3}(x+4)^{\frac{3}{2}}-10\sqrt{x+4}+c$ (B)$\frac{2}{3}(x+4)^{\frac{3}{2}}+10\sqrt{x+4}+c$ (C)$\frac{2}{3}(x+4)^{\frac{3}{2}}+\sqrt{x+4}+c$ (D)$-\frac{2}{3}(x+4)^{\frac{3}{2}}+\sqrt{x+4}+c$
3.	$\int \frac{e^x}{x+1}[1+(x+1)\log(x+1)]\, dx$ *is equal to* (A)$\frac{e^x}{x+1}+c$ (B)$\frac{e^x.x}{x+1}+c$ (C)$e^x\log(x+1)+e^x+c$ (D)$e^x\log(x+1)+c$
4.	$\int\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$ *is equal to* (A)$\frac{e^{2x}}{2}+c$ (B)$\frac{e^{2x}}{2x}+c$ (C)$\frac{e^x}{2}+c$ (D)$\frac{e^x}{2x}+c$
5.	If $\frac{1}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}$, *then the values of A and B* respectively are (A) $\frac{1}{4},\frac{1}{4}$ (B) $-\frac{1}{4},-\frac{1}{4}$ (C) $\frac{1}{4},-\frac{1}{4}$ (D) $-\frac{1}{4},\frac{1}{4}$
6.	$\int e^{4-5x}dx$ *is equal to* (A) $e^{4-5x}+c$ (B) $\frac{e^{4-5x}}{4}+c$ (C) $\frac{e^{4-5x}}{5}+c$ (D) $\frac{e^{4-5x}}{-5}+c$

7.	$\int \frac{1}{\sqrt{(x+2)^2+1}}\, dx$ is equal to (A) $\log\left\|x+\sqrt{(x+2)^2+1}\right\|+c$ (B) $\log\left\|x+2+\sqrt{(x+2)^2+1}\right\|+c$ (C)$\log\left\|x-\sqrt{(x+2)^2+1}\right\|+c$ (D) $\log\left\|x+2-\sqrt{(x+2)^2+1}\right\|+c$
8.	$\int \sqrt{x^2+a^2}dx$ is equal to (A) $\frac{x}{2}\sqrt{x^2+a^2}+\log\left\|x+\sqrt{x^2+a^2}\right\|+c$ (B)$\frac{x}{a}\sqrt{x^2+a^2}+\log\left\|x+\sqrt{x^2+a^2}\right\|+c$ (C)$\frac{x}{a}\sqrt{x^2+a^2}-\log\left\|x\sqrt{x^2+a^2}\right\|+c$ (D)$\frac{x}{2}\sqrt{x^2+a^2}-\log\left\|x+\sqrt{x^2+a^2}\right\|+c$
9.	$\int_{-2}^{2} x^5 dx$ is equal to (A) $2\int_0^2 x^5 dx$ (B)$\frac{32}{5}$ (C) $\frac{64}{5}$ (D) 0
10.	In demand curve the X- axis and Y-xis are (A) Demand, Quantity (B) Demand , Price (C) Price, Quantity (D) Demand, Supply

Answers:

1.A	2.A	3.D	4.B	5.D
6.D	7.B	8.A	9.D	10.C

Assertion and Reasoning Questions

Instructions: Answer the following questions with the following options.

(A) Both (A) and (R) are true and (R) is the correct explanation of (A)

(B) Both (A) and (R) are true and (R) is not the correct explanation of (A)

(C) (A) is true and (R) is false

(D) (A) is false, but (R) is true.

(E) Both (A) and (R) are false.

1.	Assertion(A): $\int x^2.e^x dx = x^2.e^x + e^x.2x + c$ Reason(R): $\int f(x).g(x)dx = f(x)\int g(x)dx + g(x)\int f(x)dx + c$

2.	Assertion(A): $\int e^x \left[\frac{1}{x} + logx\right] dx = e^x. logx + c$ Reason(R): $\int e^x [f(x) + f'(x)]dx = e^x. f(x) + c$
3.	Assertion(A): $\int_{-3}^{3} x^4 dx = 0$ Reason(R): $\int_{-a}^{a} f(x)dx = 0\ , if\ f(x)\ is\ an\ odd\ function.$
4.	Assertion(A): $\int_{-3}^{3} \lvert x\rvert dx = 2\int_0^3 \lvert x\rvert dx$ Reason(R): $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx\ \ , if\ f(x) is\ even\ function.$
5.	Assertion(A): $\int_1^3 x^2 dx = \frac{26}{3}$ Reason(R): $\int\ \ f(x)dx = F(x) + c$, then $\int_a^b f(x)dx = F(a) - F(b)$
6.	Assertion (A): $\int \frac{2x+3}{(x-1)(x-2)^2} dx = \int \frac{A}{x-1} dx + \int \frac{B}{(x-2)^2} dx$ Reason (R) : $\int \frac{px+q}{(x-a)(x-b)^2} dx = \int \frac{A}{x-a} dx + \int \frac{B}{(x-b)^2} dx$
7.	Assertion (A): The supply function for a commodity is $p = x^2 + 4x + 5$ where x denotes supply. The producers' surplus when the price is 10 is $10 - \int_0^2 x^2 + 4x + 5. dx$ $Reason(R): Producers\ surplus = p_o x_o - \int_0^{x_o} pdx, where\ p\ is\ the\ function\ of\ commodity$ $in\ terms\ of\ x.$
8.	Assertion(A): The consumers surplus for the demand function $p = 25 - x - x^2$ when $p_0 = 19$ is $\int_0^2 (25 - x - x^2)dx - 19.$ Reason(R): $Consumers\ surplus = \int_0^{x_o} pdx - p_o x_o, where\ p\ is\ the\ function\ of\ commodity$ $in\ terms\ of\ x.$
9.	Assertion(A) (A): $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\ dx = 3$ Reason(R): $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$
10.	Assertion (A): $\int_2^3 (x^2 + 5x + 3)dx = \int_2^3 (3 + 5t + t^2)dt$ Reason (R): $\int_a^b f(x)dx = \int_a^b f(t)dt$

Answers:

1.E	**2.B**	**3.D**	**4.B**	**5.C**
6.E	**7.D**	**8.D**	**9.A**	**10.B**

Practice:

1.	Assertion (A): $\int \frac{1}{x^2} dx = \log\lvert x^2\rvert + c$ Reason (R) : $\int \frac{1}{x} dx = \log\lvert x\rvert + c$
2.	Assertion (A): $\int_{-1}^{1} x^3 dx = 0$ Reason (R) : $\int_{-1}^{1} f(x)dx = 0$
3.	Assertion (A): $\int f(x)dx = x^3 + c, then\ f(x) = 3x^2$ Reason (R) : $\frac{d}{dx}(x^3 + c) = 3x^2$
4.	Assertion(A) (A): $\int_3^8 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{11-x}} dx = 5$ Reason(R): $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
5.	Assertion (A): $\int x^2 . logx\ dx = x^2 . \int logxdx + logx \int x^2 dx$ Reason (R) :$\int f(x). g(x)dx = f(x). \int g(x)dx + g(x) \int f(x)dx$

Answers:

1.D	**2.C**	**3.B**	**4.D**	**5.E**

Short Answer Type Questions

1. Evaluate: $\int \frac{x^3-x^2+x-1}{x-1} dx$.

Solution: $\int \frac{x^3-x^2+x-1}{x-1} dx = \int \frac{x^2(x-1)+x-1}{x-1} dx$

$$= \int \frac{(x-1)(x^2+1)}{x-1} dx = \int (x^2+1) \frac{(x-1)}{x-1} dx$$

$$=\frac{x^3}{2}+x+c$$

2. Evaluate: $\int(1-x)\sqrt{x}\,dx.$

Solution: $\int(1-x)\sqrt{x}\,dx=\int\sqrt{x}-x\sqrt{x}\ \ dx$

$$=\int x^{1/2}-x^{3/2}\ \ dx$$
$$=\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}+c$$

3. Evaluate: $\int\frac{x^2}{1+x^3}dx.$

Solution: $\int\frac{x^2}{1+x^3}dx=\frac{1}{2}\int\frac{2x^2}{1+x^3}dx$

$$\int\frac{f'(x)}{f(x)}dx.=\log|f(x)|+c.$$

$$=\frac{1}{2}\ log|1+x^3|+c$$

4. Evaluate: $\int\frac{x^3-1}{x^2}dx.$

Solution: $\int\frac{x^3-1}{x^2}dx=\int x-\frac{1}{x^2}dx$

$$=\frac{x^2}{2}+\frac{1}{x}+c$$

5. Evaluate: $\int\frac{(logx)^2}{x}dx.$

Solution: $\int\frac{(logx)^2}{x}dx=\int t^2\,dt$ where take $logx=t$

$$=\frac{t^3}{3}+c$$

$$=\frac{(logx)^3}{3}+c$$

6. Evaluate: $\int\frac{1}{x^2+16}dx.$

Solution: $\int\frac{1}{x^2+16}dx=\int\frac{1}{x^2+4^2}dx$

$$\int\frac{1}{a^2+x^2}dx.=\frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)+c.$$
$$=\frac{1}{4}tan^{-1}\left(\frac{x}{4}\right)+c.$$

7. Evaluate: $\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{\sqrt{1-x^2}}dx.$

Solution: $\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}} dx = [sin^{-1}x]_0^{\frac{1}{\sqrt{2}}}$

$$= sin^{-1}\frac{1}{\sqrt{2}} - sin^{-1}0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

8. If $\int_0^1 (3x^2 + 2x + k)dx = 0$, find the value of k.

Solution: given $\int_0^1 (3x^2 + 2x + k)dx = 0$

$$\left[\frac{3x^3}{3} + \frac{2x^2}{2} + kx\right]_0^1 = 0$$

$$(1 + 1 + k) - 0 = 0$$

$$k = -2$$

9. Evaluate: $\int_0^2 \frac{1}{\sqrt{4+x^2}} dx$.

Solution: $\int_0^2 \frac{1}{\sqrt{4+x^2}} dx = \int_0^2 \frac{1}{\sqrt{2^2+x^2}} dx$

$\int \frac{1}{\sqrt{a^2+x^2}} dx.$ = log $|x + \sqrt{a^2 + x^2}| + c$

$$= \left[\log|x + \sqrt{2^2 + x^2}|\right]_0^2$$
$$= \log\left|2 + \sqrt{2^2 + 2^2}\right| - \log|0 + \sqrt{2^2}|$$
$$= \log\left|2 + 2\sqrt{2}\right| - \log|2|$$
$$= \log\left|1 + \sqrt{2}\right|$$

10. Evaluate: $\int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx$.

Solution: $\int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx = \int_{\frac{1}{3}}^1 \frac{x(\frac{1}{x^2}-1)^{\frac{1}{3}}}{x^4} dx$

$$= \int_{\frac{1}{3}}^1 \frac{x(\frac{1}{x^2}-1)^{\frac{1}{3}}}{x^4} dx$$

$$= \int_{\frac{1}{3}}^1 \frac{x(\frac{1}{x^2}-1)^{\frac{1}{3}}}{x^4} dx$$

$$= \int_{\frac{1}{3}}^1 \frac{(\frac{1}{x^2}-1)^{\frac{1}{3}}}{x^3} dx$$ where take $\frac{1}{x^2} - 1 = t$

$$= \int_8^0 t^{\frac{1}{3}} \frac{dt}{-2} = \frac{-1}{2} . \frac{3}{4}\left[t^{\frac{4}{3}}\right]_8^0$$

$$= \frac{-1}{2} . \frac{3}{4} \left[0 - 8^{\frac{4}{3}}\right] \quad = \frac{-1}{2} . \frac{3}{4} .16 = 6$$

Practice problems

Very short Answer type Questions:

1. Evaluate: $\int 5^x \, dx$.
2. Evaluate: $\int_1^3 (x^2 + 2x) \, dx$.
3. Evaluate: $\int_2^3 \frac{1}{x} dx$.
4. Evaluate: $\int_0^2 \frac{1}{\sqrt{4-x^2}} dx$.
5. Evaluate: $\int_0^1 \frac{1}{1+x^2} dx$.
6. If $\int_0^a \frac{1}{4+x^2} dx = \frac{\pi}{8}$, find the value of a.
7. Evaluate $\int_0^2 (x^2 + e^{2x+1}) dx$.
8. Evaluate: $\int_0^1 \frac{2x}{1+x^2} dx$.
9. Evaluate: $\int \frac{1}{xlogx} \, dx$.
10. Evaluate: $\int \frac{x}{1+x^2} \, dx$.

Answers:

1. $5^x log5$
2. $\frac{50}{3}$
3. $log \frac{3}{2}$
4. $\frac{\pi}{2}$
5. $\frac{\pi}{4}$
6. 2
7. $\frac{8}{3} + \frac{e^5}{2} - \frac{e}{2}$
8. $log2$
9. $\log(logx)$
10. $\frac{1}{2}\log(1 + x^2)$

Short answer questions:

1. Evaluate: $\int \frac{1}{x(x^5+3)} dx$.
2. Evaluate: $\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$.
3. Evaluate: $\int x. log2x \, dx$.
4. Evaluate : $\int \frac{x-3}{(x-1)^3} . e^x dx$
5. Evaluate: $\int e^x \frac{(x^2+1)}{(x+1)^2} dx$.
6. Evaluate: $\int_{-1}^2 |x^3 - x| \, dx$.
7. Evaluate: $\int_2^8 |x - 5| \, dx$.
8. Evaluate: $\int_1^4 [|x - 1| + |x - 2| + |x - 3|] \, dx$.
9. Find: $\int \frac{e^x}{(e^x-1)\,(e^x+2)} dx$.
10. Evaluate: $\int \frac{1}{5-8x-x^2} dx$.

Answers:

1. $\frac{1}{15} log\left|\frac{x^5}{(x^5+3)}\right| + c$ 2. $\sqrt{x^2 + 2x + 3} + log\left|x + 1 + \sqrt{x^2 + 2x + 3}\right| + c3.\frac{x^2}{2} log2x - \frac{x^2}{4} + c$

4. $\frac{e^x}{(x-1)^2} + c$ 5. $e^x \left(\frac{x-1}{x+1}\right) + c$ 6. $\frac{11}{4}$ 7. 9 8. $\frac{19}{2}$

9. $\frac{1}{3} log\left|\frac{(e^x-1)}{(e^x+2)}\right| + c$ 10. $log\left|\frac{x+2-\sqrt{21}}{x+2+\sqrt{21}}\right|$

Applications of the integrals

Practice Problems:

1. Find the consumers' surplus for the demand function p = 25 - x – x^2 when p_0 =19.
2. The demand function for a commodity is $P = \frac{10}{x+1}$. Find the consumers' surplus when the prevailing market price is 5.
3. The supply function for a commodity is p = x^2 + 4x + 5 where x denotes supply. Find the producers' surplus when the price is10.
4. If the demand function for a commodity is p = 25- x^2, find the consumers' surplus for p_0 =9.
5. If the supply function is p = $3x^2$ + 10 and x_0 = 4, then find the producers'surplus.
6. The marginal cost of production of x units of a commodity is 30+2x. It is known that fixed costs are Rs.120. Find the total cost of producing 100units.
7. The marginal revenue function of a commodity is MR = $7 - \frac{6}{(x+2)^2}$. find the revenue function. Also find the revenue obtained on selling 4 units of the product.
8. Determine the cost of increasing output from 100 to 200 units if the marginal cost MC = $0.003x^2$-0.01x+2.5.
9. A manufacturer's marginal cost function is $\frac{500}{\sqrt{2x+25}}$. Find the cost involved to increase production from 100 units to 300 units.
10. The marginal revenue function of a commodity is MR = $9 + 6x^2$, find the revenue function. Also, find the revenue obtained on selling 5 units of the product.

Answers:

1. $\frac{22}{3}$ 2. $10log2 - 5$ 3. $\frac{8}{3}$ 4. $\frac{128}{3}$ 5. 128

6. $C(x) = 120 + 30x + x^2$ 7. $R(x) = 7x + \frac{6}{x+2}$ 8. Rs. 7100

9. $500\sqrt{2x + 25}$ 10. $R(x) = 9x + 2x^3$

LONG ANSWER TYPE QUESTIONS WITH SOLUTION

Q1. Find $\int \frac{3x-2}{(x+1)(x-2)^2}$dx

Solution:

Let $\frac{3x-2}{(x+1)(x-2)^2} = \frac{A}{(X+1)} + \frac{B}{(X-2)} + \frac{C}{(X-2)^2}$

$3x – 2 = A(x – 2)^2 + B(x + 1)(x – 2) + C(x + 1)$

$3x – 2 = A(x^2 – 4x + 4) + B(x^2 – x – 2) + C(x + 1)$

$= (A + B) x^2 + (–4A – B + C) x + 4A – 2B + C$

Comparing coefficients of x^2, x and constant terms on both sides, we get

A + B = 0, –4A – B + C = 3, 4A – 2B + C = –2

Solving we get A= -5/9 ,B =5/9 C= 4/3

$I = \int \frac{3x-2}{(x+1)(x-2)^2} dx = \frac{-5}{9} \int \frac{1}{x+1} dx + \frac{5}{9} \int \frac{1}{x-2} dx + \frac{4}{3} \int \frac{1}{(x-2^2)} dx$

$I = \frac{-5}{9} \log|x+1| + \frac{5}{9} \log|x-2| - \frac{4}{3(x-2)} + C$

Q2. Find $\int \frac{(x-1)(x-2)}{(x-3)(x-4)}$ **dx**

Solution:

$\frac{(x^2-3x+2)}{x^2-7x+12)} = \frac{(x^2-7x+12+4x-10)}{(x^2-7x+12+4x)}$

$= 1 + \frac{4x-10}{(x-3)(x-4)}$

Now $\frac{4x-10}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$

4x–10 = A(x–4) + B(x–3) = (A + B)x + (–4A – 3B)

A + B = 4, –4A – 3B = -10

Solving we get A = -2, B =6

$I = \int \frac{3x-2}{(x+1)(x-2)^2} dx = \int \left[1 + \frac{-2}{(x-3)} + \frac{6}{(x-4)}\right] dx$

$= x-2\log|x-3| + 6\log|x-4| + C$

Q3.Find $\int \frac{x^3+x+1}{x^2-1}$dx

Solution

$\frac{x^3+x+1}{x^2-1} = 1 + \frac{2x+1}{x^2-1}$

$$I \; = \int x dx + \int \frac{2x+1}{x^2 - 1} dx$$

$$= \int x dx + \int \frac{2x}{x^2 - 1} dx \quad + \int \frac{1}{x^2 - 1} dx$$

$$= \frac{x^2}{2} \; + \; \log (x^2 - 1) + \frac{1}{2} \log\left|\frac{1-x}{1+x}\right| + c$$

Q4. Evaluate $\int_1^5 (|x - 1| + |x - 2| + |x - 3|)dx$

Solution

$$\int_1^5 (|x - 1| + |x - 2| + |x - 3|) \, dx$$

$$= \int_1^5 (x - 1)dx + \int_1^2 (2 - x) \, dx + \int_2^5 (x - 2) \, dx + \int_1^3 (3 - x)dx + \int_3^5 (x - 3) \, dx$$

$$= \; \left[\frac{(x-1)^2}{2}\right]_1^5 \; + \left[\frac{(2-x)^2}{-2}\right]_1^2 \; + \left[\frac{(x-2)^2}{2}\right]_2^5 + \left[\frac{((3-x)^2}{-2}\right]_1^3 + \left[\frac{(x-3)^2}{2}\right]_3^5$$

$$= \; \frac{16}{2} + \; 0 - (\frac{1}{-2} \;) \; + \frac{9}{2} + 0 - (\frac{4}{-2}) \; + \frac{4}{2}$$

$$= \; 17$$

Q5.The demand and supply functions under the pure market competition are $p_d = 16 - x^2$ and $p_s = 2x^2 + 4$ respectively,where p is the price and x is the quantity of the commodity.

(i) Find the price p_0 and quantity x_0

(ii) Using integrals find Consumer's surplus.

(iii) Using integrals find Producers'surplus.

Solution:

Under pure market competition

$p_d = p_s$

$\Longrightarrow 16 - x^2 = 2x^2 + 4$

$\Longrightarrow 3x \; 2 = 12 \Longrightarrow$ x = 2, -2; since x can't be –ve, so x=2

When $x_0 = 2$; $p_0 = 12$

Consumer's surplus $= \int_0^2 P_d dx - p_0 x_0$

$2\int_0^2 (16 - x^2)dx - 12 \times 2 = 16/3$ units

Producers' surplus$= p_0 x_0 - \int_0^2 p_s dx = 24 - \int_0^2 (2x^2 + 4)dx = \frac{32}{3}$units

LONG ANSWER TYPE QUESTIONS (EXERCISE)

1. The demand and supply functions under the pure market competition are $pd = 56 - x^2$ and $ps = 8 + \frac{x^3}{3}$respectively, where p is the price and x is the quantity of the commodity.
    - i. Find the price p_0 and quantity x_0
    - ii. Using integrals, find the Consumer's surplus.
    - iii. Using integrals find Producers' surplus

2\. Find $\int \frac{x^3 + x^2 + x + 1}{x + 1}$

3\. Find $\int \frac{5x+4}{(x^2 - 1)(x+2)}$ dx

4\. Find $\int \frac{x}{(x+2)(x-1)^2}$ dx

5\. The marginal revenue function for a firm is given by $\frac{5x^2 + 30x + 51}{(x+3)^2}$

Show that the revenue function is given by $\frac{2x}{x+3}$ + 5x

6\. Evaluate the following definite integrals.

- i. $\int_0^1 \frac{3t^2}{(1+t^3)(2+t^3)}$ dt
- ii. $\int_{\mathbf{0}}^{\mathbf{4}} (|\boldsymbol{x}| + |\boldsymbol{x} - \mathbf{2}| + |\boldsymbol{x} - \mathbf{4}|)$dx
- iii. $\int_{-\mathbf{1}}^{\mathbf{1}} \frac{\boldsymbol{x}^{\mathbf{3}} + |\boldsymbol{x}| + \mathbf{1}}{\boldsymbol{x}^{\mathbf{2}} + \mathbf{2}|\boldsymbol{x}| + \mathbf{1}}$dx

ANSWERS

1)

- i. $x_6 = 6$, $p_{0\,=}\, 20$
- ii. CS = 264
- iii. PS = 48 Units

2) $\frac{x^3}{3}$ + x + c

3) $\frac{3}{2}$ log $|x - 1|$+ $\frac{1}{2}$ log $|x + 1|$ - 2 log $|x + 2|$ + c

4) $\frac{1}{2}$ log $|x - 2|$ - $\frac{1}{3(x-1)}$ - $\frac{2}{9}$ log $|x + 2|$ + c

5) $\frac{2x}{x+3}$ + 5x

6) i) $\log\frac{4}{3}$

ii) 20

iii) 2log2

CASE BASED QUESTIONS WITH SOLUTION

Q1. Boby is taking a learning test in which the time he takes to memorize items from a given list is recorded. Let M(t) be the number of items he can memorize in t minutes. His learning rate is found to be M(t) = 0.4t – 0.005t^2

i) How many items can Boby memorize in t seconds?

ii) How many items can Boby memorize during the first 10 minutes?

iii) How many additional items can he memorize during the next 10 minutes?(From time t= 10 to t =20)

SOLUTION:

(i) M(t) = 0.4t – 0.005t^2 dt

$M(t) = \int M'(t)dt = \int(0.4t - 0.005t^2)dt$

$M(t) = 0.4\frac{t^2}{2} - .005\frac{t^3}{3} + c$

If t = 0 , M(t) = 0

0 = 0-0+c

c =0

(i) $M(t) = 0.2t^2 - 0.005\frac{t^3}{3}$

(ii) $M(10) = 0.2(10^2) - 0.005\left(\frac{10^3}{3}\right) = \frac{55}{3}$ = 18.33 item

(iii) $M(20) - M(10) = 0.2(20^2) - 0.005\left(\frac{20^3}{3}\right) - \frac{55}{3}$

$= \frac{145}{3} =$ 48.33 item

Q2. A tyre manufacturer estimates that (thousand) radial tyres will be purchased i.e. demanded by whole sales when price is p= D(x) =90 - $\frac{x^2}{10}$ thousand rupees per tyre and the same number of tyres will be supplied when the price is

p = S(x) = $\frac{1}{5}x^2$+x+50 thousand rupee per tyre.

i) Find the equilibrium price and the quantity supplied and demanded at that price.

ii) Determine the consumers and producer's surplus at the equilibrium price.

Solution:

The equilibrium point ($x_{0,}$ p_0) is the point at which the demand and supply curves intersect. Therefore the equilibrium point is obtained by setting D(x) = S(x)

$\Longrightarrow 90 - \frac{x^2}{10} = \frac{x^2}{5} + x + 50$

$\Longrightarrow 3x^2 + 10x - 400 = 0$

$\Rightarrow$ (x -10)(3x + 40) = 0

$\Rightarrow$ x =10

Putting x= 10 in D(x), we get p =80 x = 10

(i) x = 10 ⇒No. of tyres = 10×1000= 10,000, Price =Rs 80, 000

(ii) CS = $\int_0^{x_0} D(x)dx - p_0 x_0$

CS = $\int_0^{10} \left[90 - \frac{x^2}{10}\right]dx - 80 \times 10 =$

$\Rightarrow$ CS = $\left[900 - \frac{1000}{30}\right] - 800 = \frac{200}{3} = 66.667$

Since x is in thousands, CS = 66667

(ii) The producer’s surplus (PS) is given by

PS = $p_0x_0 - \int_0^{x_0} S(x)dx$

$\Rightarrow$ PS = $80 \times 10 - \int_0^{10} \left[\frac{1}{5}x^2 + x + 50\right] dx$

$\Rightarrow$ PS = $800 - \left[\frac{x^3}{15} + \frac{x^2}{2} + 50x\right]_0^{10}$

$\Rightarrow$ PS = $800 - \left[\frac{1000}{15} + \frac{100}{2} + 500\right] = 800 - \left[\frac{200}{3} + 50 + 500\right] = \frac{550}{3} = 183.33$

CASE BASED QUESTIONS (Exercise)

Q1. The second new species named puntius euspilurus is an edible fresh water fish found in the mananthavady river in wayanad. The epithet euspilurus is a greek word referring to the distinct black spot on the caudal fin. The slender bodied fish prefers fast flowing, shallow and clear waters and occurs only in unpolluted areas. It appears in great numbers in paddy fields during the onset of the southwest monsoon.

Suppose that the supply schedule of this fish is given in the table below which follows a linear relationship between price and quantity supplied.

PRICE P PER KG (IN Rs)	QUANTITY (X) OF FISH SUPPLIED (IN KG)

25	800
20	700
15	600
10	500
5	400

Suppose that this fish can be sold only in the Kerala. The Kerala demand schedule for this fish is as follows and there in a linear relationship between price and quantity demanded.

PRICE P PER KG (IN Rs)	QUANTITY (X) OF FISH SUPPLIED (IN KG)
25	200
20	400
15	600
10	800
5	1000

i) Which of the following represents the price (P) supply(x) relationship?

a) p= 65 - $\frac{X}{20}$ b) p= 65+ $\frac{X}{20}$

c) p= -15 + $\frac{X}{20}$ d) p= 15 - $\frac{X}{20}$

ii)The equation of demand curve can be given by

a) p= 30 - $\frac{X}{40}$ b) p= 30 + $\frac{X}{40}$

c) p= 20 - $\frac{X}{40}$ d) p= 20 + $\frac{X}{40}$

iii)The value of x at equilibrium is

a) 1400/3 b) 600

c) 15 d) 200/3

iv)The equilibrium price is

a) 400 b) 20

c) 600 d) 15

v)The consumer's surplus at equilibrium price is

a) 18009 b) 13500

c) 9000 d) 4500

Q2. A firm finds that quantity demanded and quantity supplied is 30 units when market price is Rs 8 per unit. Further, if price is increased to Rs 12 per unit, demand reduces to 0 and at a price of Rs 5 per unit, the firm is not willing to produce. Assuming the linear relationship between price and quantity in both cases,

i) Find the demand function

ii) Supply function

iii)Consumers surplus and producers surplus at equilibrium price.

ANSWERS

1. i)c ii)a iii) b iv) d v) d

2. i) Demand Function : $p= 12 - \frac{2x}{15}$

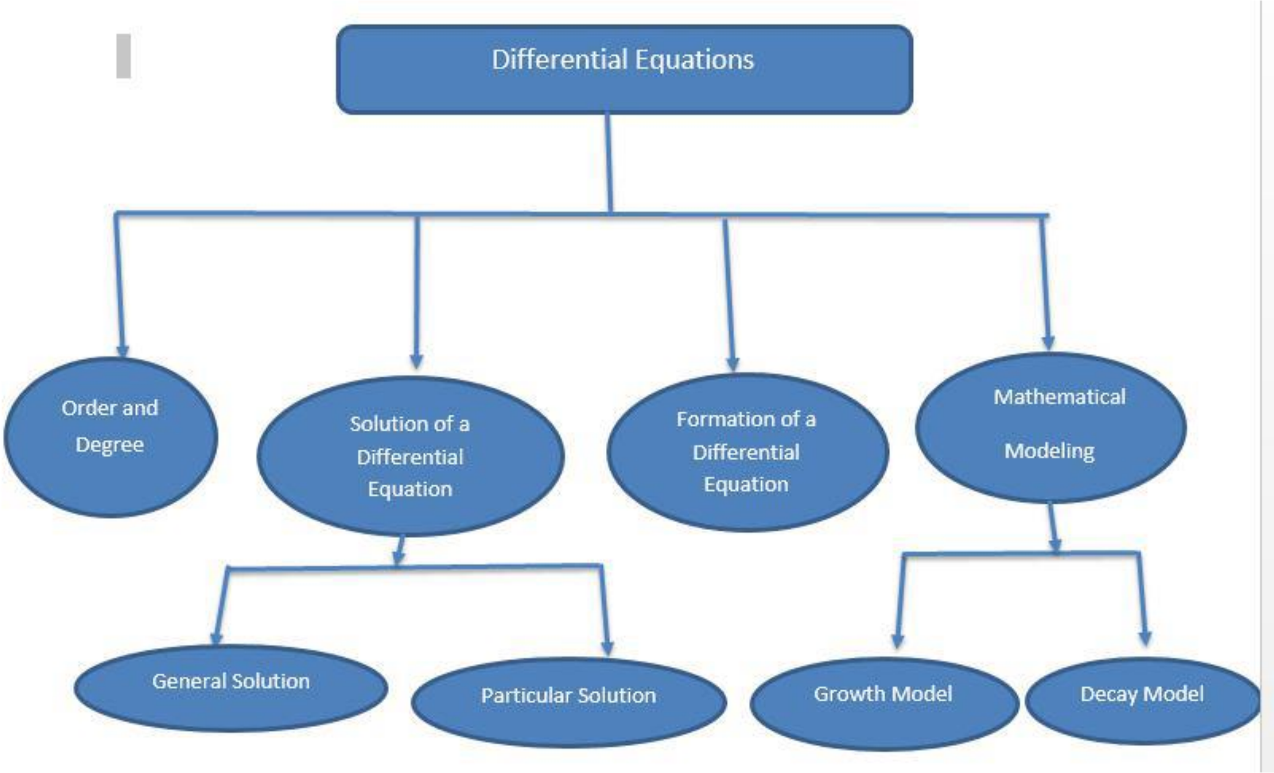
ii)Supply function : $p = \frac{x}{10} +5$

iii)Consumers' Surplus = 60

Producers surplus = 45

UNIT 3C- DIFFERENTIAL EQUATIONS AND MODELLING

CONCEPT MAP



Differential Equation: Let y = f(x). An equation of the form $\frac{dy}{dx} = g(x)$ or $f'(x) = g(x)$ is known as differential equation.
In general,
An equation involving derivative(s) of the dependent variable with respect to the independent variable (s) is called a differential equation.

A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation.

Examples:

1. $\frac{dy}{dx} + \frac{y}{x} = x^3$

2. $\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0$

3. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

4. $\frac{d^2y}{dx^2} + y^2 + e^{\frac{dy}{dx}} = 0$

5. $\left(\frac{d^3y}{dx^3}\right)^2 - 3\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^4 = y^4$

Order of a Differential Equation

The order of differential equation is defined as the highest ordered derivative of the dependent variable with respect to the independent variable involved in the differential equation.

The differential equations (1), (2), (3), (4) and (5) mentioned earlier involve the highest derivative of first, third, second, second and third order respectively. Therefore the order of these differential equations is 1, 3, 2, 2 and 3 respectively.

Degree of a Differential Equation

The degree of a differential equation, when it is a polynomial equation in its derivatives is the highest power (positive integral index) of the highest order derivative involved in the differential equation.

We observe that differential equations (1), (2), (3) and (5) are polynomial equations in its derivatives therefore their degrees are defined. But equation (4) is not a polynomial equation in $\frac{dy}{dx}$, therefore its degree is not defined.

In view of the above definition, the differential equation (1), (2), (3) and (5) have degrees 1, 1, 1 and 2 respectively.

General and particular solutions of a Differential Equation

General Solution: The function involving the variables and independent arbitrary constants is called the general solution of the differential equation.

A general solution of the differential equation in two variables x and y and having two arbitrary constants c_1, c_2 is of the form $f(x, y, c_1, c_2) = 0$.

Particular Solution : A solution obtained from the general solution by giving particular values to arbitrary constants is called a particular solution of the differential equation. A particular solution in two variable x and y not contain any arbitrary constant and is of the form f(x, y) = 0.

Formation of a Differential Equation

Steps to form a Differential Equation

Step 1: Let us assume the given family of curves ‘f’ depends on the parameters a, b (say), then it is represented by an equation of the form f(x, y, a, b) = 0 → (1)

Step 2: Differentiate equation (1) with respect to ‘x’, we getg(x, y, y’, a, b) = 0 → (2)

Step 3: But it is not possible to eliminate two parameters ‘a’ and ‘b’ from two equations. So, a third equation is obtained by differentiating equation (2) with respect to ‘x’ to obtain a relation of the form h(x, y, y’, y’’,a, b) = 0 → (3)

Step 4: The required differential equation is obtained by eliminating ‘a’ and ‘b’ from equations (1),(2) and (3) to get F(x, y, y’, y’’) = 0.

Note: If the given family of curves has *n* parameters then it is to be differentiated *n* times to eliminate the parameters and obtain the nth order differential equation.

Solving simple differential equation

Type 1: An equation of the form $\frac{dy}{dx} = F(x)$(RHS is function of ‘x’ alone)

Given Differential Equation $\frac{dy}{dx} = F(x)$

Separate the variables as dy = F(x)dx

Integrating both sides, we get $\int 1\, dy = \int F(x)dx$

y = $\int F(x)dx$ + c, where ‘c’ is arbitrary constant

Type 2: An equation of the form $\frac{dy}{dx} = G(y)$(RHS is function of ‘y’ alone)

Given Differential Equation $\frac{dy}{dx} = G(y)$

Separate the variables as$\frac{dy}{G(y)} = dx$

Integrating both sides, we get $\int \frac{1}{G(y)} dy = \int 1\, dx$

$\int \frac{1}{G(y)} dy$ = x + c, where ‘c’ is arbitrary constant

Type 3: An equation of the form: $\frac{dy}{dx} = f(x, y)$(RHS is a product of function of ‘x’ alone and function of ‘y’ alone)

Given Differential Equation $\frac{dy}{dx} = F(x,\ y)$where F(x, y) = f(x).g(y)

$$\frac{dy}{dx} = f(x).\, g(y)$$

Separate the variables as $\frac{dy}{g(y)} = f(x)dx$

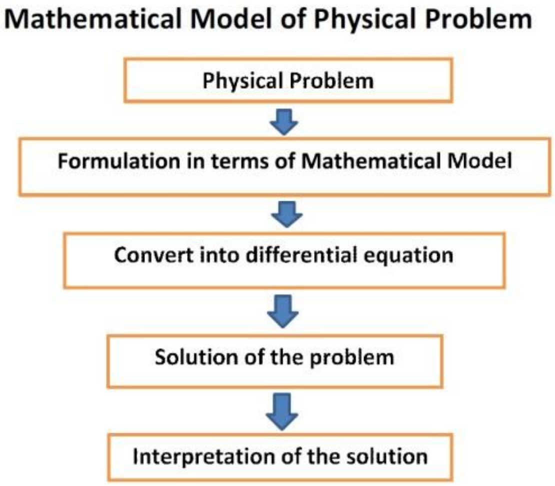
Integrating both sides, we get

$\int \frac{1}{g(y)} dy = \int$ f(x) dx + c, where ‘c’ is arbitrary constant

Differential Equations and Mathematical Modeling

The steps to be followed while solving mathematical modelling are as follows:

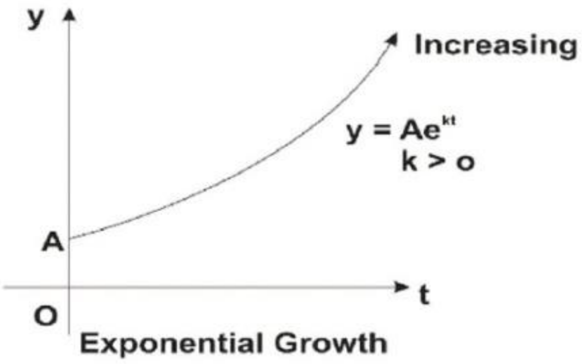
1. The formulation of a real-world problem in mathematical terms, i.e., the construction of a mathematical model.
2. The solution of the mathematical problem.
3. The interpretation of the mathematical result in the context of the original problem.

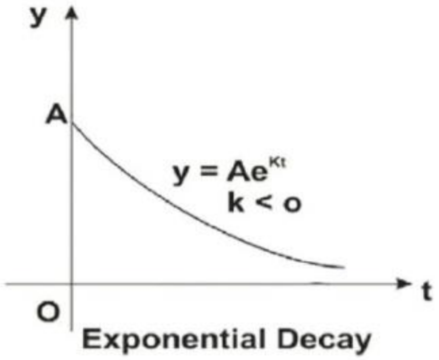


Growth and Decay Models

The mathematical model for exponential growth or decay is given by $f(t) = Ae^{kt}(or)\ y = \ Ae^{kt}$

Where:	*t*	represents time
	A	the original amount
	y or *f*(*t*)	represents the quantity at time *t*
	k	is a constant that depends on the rate of growth or decay
	If $k > 0$,	the formula represents exponential growth
	If $k < 0$,	the formula represents exponential decay





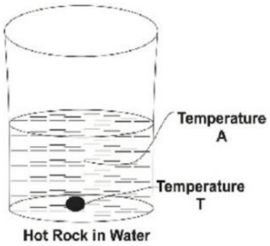
Population Growth : Suppose that P(t) is the number of individuals in a population (of humans or insects or bacteria) having constant birth rate $\alpha$and constant death rate β.

Then$P(t) = Ae^{kt}$, for all real ‘t’, where $k = \ \alpha - \beta$ and ‘A’ is arbitrary constant.

Compound Interest: A person deposits an amount A(t) at a time ‘t’ (in years) in a bank and suppose that the interest is compounded continuously at an annual interest rate ‘r’.

Then$A(t) = A_0e^{rt}$, where A_0 is the money deposited at t = 0.

Newton's Law of Cooling: The rate of change of temperature T of a body is proportional to the difference between T and the temperature of the surrounding medium A.



$\frac{dT}{dt} = -k(T - A)$, *where k > 0.*

If $T > A$, *then* $\frac{dT}{dt} < 0$

Temperature of the body is a decreasing function of time and the body is cooling.

If $T < A$, *then* $\frac{dT}{dt} > 0$

Temperature of the body is an increasing function of time and the body is heating.

Remark:

1. The physical law is translated into a differential equation
2. If value of *k* and *A* are known, we can determine the temperature *T* of the body at any time *t*.

Carbon Dating

Let A(t) be the mass of carbon-14 after 't' years.

$$A = A_0 e^{-kt} \qquad (\because A = A_0 \text{ when } t = 0)$$

MULTIPLE CHOICE QUESTIONS

1.	Order of differential equation corresponding to family of curves $y = Ae^{2x} + Be^{-2x}$ is (a) 2 (b) 1 (c) 3 (d)4
2.	The order of the differential equation corresponding to the family of curves $y=c(x-c)^2$, where c is constant (a) 1 (b) 2 (c) 3 (d)does not exist
3.	General solution of the differential equation $\log\frac{dy}{dx} = 2x + y$ is (a) $e^{-y} = \frac{1}{2}e^{2x} + C$ (b) $\frac{1}{e^x} + \frac{1}{2}e^{2x} + C$ (c) $-e^{-y} = \frac{1}{2}e^{2x} + C$ (d) $e^{y} = \frac{1}{2}e^{2x} + C$
4.	The particular solution of the differential equation $\frac{dy}{dx} = y \tan x$, given that y = 1 when x = 0 is dx (a) y = cos x (b) y = sec x (c). y = tan x (d) y = sec x tan x
5.	Differential equation representing the family of curves $(x+ a)^2 + 2y^2 = a^2$ is of order

	(a) 1 (b) 2 (c) 3 (d) none of these
6.	Y $=e^{-x}+ax+b$ is a solution of differential equation (a) $e^{-x}\, y'' = 1$ (b) $e^{x}\, y'' = 1$ (c) $e^{x}\, (y')^2 = 1$ (d) $e^{-x}\, (y')^2 = 1$
7.	Degree of differential equation $t^2\frac{d^2s}{dt^2}$ - $st(\frac{ds}{dt})^2 = 5$ is (a) 1 (b) 2 (c) 3 (d)none of these
8.	Degree of differential equation $\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}} = x$ is (a) 1 (b) 2 (c) 3 (d) none of these
9.	The sum of order and degree of the differential equation $\frac{d^2y}{dx^2}+3(\frac{dy}{dx})^3=e^x$ is (a) 2 (b) 3 (c) 5 (d)4
10.	The number of arbitrary constants in the general solution of differential equation of fourth order are (a) 0 (b) 2 (c) 3 (d) 4

Answers:

1	a	2	a	3	c	4	b	5	a
6	b	7	a	8	b	9	b	10	d

PRACTICE QUESTIONS

01	The number of arbitrary constants in the particular solution of a differential equation of mth order is____________, where m is an integer. (a) m (b) 1/m (c) 0 (d) 1
02	The highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation is called____ of the differential equation. (a) homogeneous (b) power (c) degree (d) order

03	An equation involving derivatives of the dependent variable with respect to independent variables is called a/ an_____ (a) ordinary differential equation (b) partial differential equation (c) differential equation (d) linear equations
04	A solution of differential equation which contains arbitrary constants is called the ________ of the differential equation (a) solution (b) optimal solution (c) general solution (d) particular solution
05	A differential equation involving derivatives of the dependent variable with respect to only One independent variable is called a/an ________________. (a) ordinary differential equation (b) partial differential equation (c) differential equation (d) linear equations

Answers for MCQ's

1	c	2	d	3	c	4	c	5	a

ASSERTION REASONING-BASED QUESTIONS

In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not the correct explanation of A.

(c) A is true but R is false.

(d) A is false but R is true

1.	**Assertion (A) :** The degree of the differential equation given by $\frac{dy}{dx} = \frac{x^4-y^4}{(x^2+y^2)xy}$ is 1 **Reason (R) :** The degree of a differential equation is the degree of the highest order derivative when differential coefficients are free from radicals and fraction.
2.	**Assertion (A):** The order and degree of differential equation $\sqrt{\frac{d^2y}{dx^2}} = \sqrt{\frac{dy}{dx}+5}$ are 2 and 1 respectively. **Reason (R):** The differential equation $\left(\frac{dy}{dx}\right)^3 + 2y^{\frac{1}{2}} = x$ is of order 1 and degree 3.

3.	**Assertion (A):** The degree of the differential equation $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2\log(\frac{d^2y}{dx^2})$is not defined. **Reason (R):** If the differential equation is a polynomial in terms of its derivatives, then its degree is defined.
4.	**Assertion (A): order of differential equation** $\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2} = sinx$ **is 1** **Reason (R):** Order of the differential equation is the order of the highest order differential present in the equation.

Answers

Q1: (a) Explanation: The given differential equation has first order derivative which is free from radical and fraction with power = 1, thus it has a degree of 1.

Q2: (b) **Q3: (a)** **Q4: (d)**

VERY SHORT ANSWER QUESTIONS (2mark QUESTIONS)

1. Find the sum and degree of the differential equation $1 + \left(\frac{dy}{dx}\right)^2 = x$

Sol: Order of the Differential equation is 1, As the highest order of derivative is $\frac{dy}{dx}$

Degree of the differential equation is 2

Hence sum of them is 1+2 =3.

2. Find the value of *m* and *n*, where *m* and *n* are order and degree of differential equation

$$\frac{4\left(\frac{d^2y}{dx^2}\right)^3}{\frac{d^3y}{dx^3}} + \frac{d^3y}{dx^3} = x^2 - 1.$$

Sol: On simplifying we get $4\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{d^3y}{dx^3}\right)^2 = \frac{d^3y}{dx^3}(x^2 - 1)$

Hence Order of differential equation (*m*) = 3

Degree of differential equation (*n*) = 2

3. Find the general solution of differential equation: $ylogy\ dx - x\ dy = 0$

Sol: On simplifying we get $\int \frac{dy}{ylogy} = \int \frac{dx}{x}$

$$\Rightarrow \log(logy) = \log|x| + \log|c|$$

$$\Rightarrow \log(logy) = \log|cx| \qquad \Rightarrow y = e^{|cx|}$$

4. Show that $y^2 = 4ax$ is a solution of the differential equation $y = x\frac{dy}{dx} + a\frac{dx}{dy}$.

Sol: We have $y^2 = 4ax$

Differentiating both sides with respect to x we get

$$2y\frac{dy}{dx} = 4a \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y} \quad \Rightarrow x\,\frac{dy}{dx} = \frac{2ax}{y} \;\; -\!\!\rightarrow (1)$$

Also $\frac{dx}{dy} = \frac{y}{2a} \Rightarrow \mathrm{a}\frac{dx}{dy} = \frac{y}{2} \;--\!\!\rightarrow (2)$

Substituting (1) and (2) in the RHS of the given DE, we get

RHS = $x\frac{dy}{dx} + a\frac{dx}{dy} \;=\; \frac{2ax}{y} + \frac{y}{2} = \frac{4ax+y^2}{2y} = \;\frac{y^2+y^2}{2y} = \frac{2y^2}{2y} = y = LHS$

Hence $y^2 = 4ax$, is the solution of the given differential equation.

5. Form a differential equation representing the family of curves given by $y = ae^{bx}$ where a, b are arbitrary constants .

Sol: Differentiating both sides of given DE with respect to x we get

$\frac{dy}{dx} = \;ab\,e^{bx} = b.y\;\;(since\; y = ae^{bx})$ ……(1)

$\Rightarrow \mathrm{b} = \frac{1}{\mathrm{y}}\frac{\mathrm{dy}}{\mathrm{dx}}$ …..(2)

Differentiating (1) with respect to x, we get $\frac{d^2y}{dx^2} = b\frac{dy}{dx} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 \quad (using\;(2))$

Hence required differential equation is $\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2$

6. Find the general solution of the following differential equations $\frac{dy}{dx} = \frac{x+1}{2-y}$

Sol: Separating the variables on either side and integrating we get

$$\int(2-y)dy = \int(x+1)dx \quad \Rightarrow \quad 2y - \frac{y^2}{2} = \frac{x^2}{2} + x + C$$

Which is the required solution.

7. The amount of radiocarbon present after t years is given by $A = A_\circ e^{-(ln2)\left(\frac{1}{5700}\right)t}$, where is the amount present in the living plants and animals. Find the half-life of radiocarbon.

Sol: $\frac{A_\circ}{2} = A_\circ\, e^{\left(-\frac{ln2}{5700}\right)t}$

$\left(-\frac{ln2}{5700}\right)t = \ln(\frac{1}{2}) \quad \Rightarrow \quad t = \frac{-5700}{ln2}\;.-(ln2)$ = 5700 years

Hence the half time is 5700 years.

8. Form the differential equation of the family of parabolas having vertex at the origin and axis along positive y-axis.

Sol: Required equation of parabola is $x^2 = 4ay$

Differentiating with respect to x, we get $2\mathrm{x} = 4\mathrm{a}\frac{\mathrm{dy}}{\mathrm{dx}}$

$\Rightarrow \frac{x^2}{2x} = \frac{4ay}{4a\frac{dy}{dx}} \quad \Rightarrow \quad x\frac{dy}{dx} - 2y = 0$ which is the required differential equation.

9. Find the differential equation of the family of curves given by $x^2 + y^2 = 2ax$.

Sol: $x^2 + y^2 = 2ax$ ...(i)

Differentiating, we get $2x + 2yy' = 2a$,

Substituting in (i), we get,

$x^2 + y^2 = x(2x + 2yy')$ $\Rightarrow 2xyy' + x^2 - y^2 = 0$ is the required equation.

10. Verify $y = Ax$ it a solution of differential equation $x\frac{dy}{dx} = y, x \neq 0$.

Sol: Consider $y = Ax$

Differentiating with respect to x, we get $\frac{dy}{dx} = A$

Substituting in the given Differential equation we get, $LHS = x\frac{dy}{dx} = x.A = y = RHS.$

Hence $y = Ax$ is the solution of the given differential equation.

SHORT ANSWER QUESTIONS (3 MARKS)

1. In a bank Principal increases continuously at the rate of 5% per year. An amount of ₹1000 is deposited with this bank. How much will it worth after 10 years. $(e^{0.5} = 1.648)$

Sol: As per the given data $\frac{dP}{dt} = \frac{5}{100}P \Rightarrow \frac{dp}{p} = \frac{1}{20}t$ Integrating we obtain

$$logP = \frac{1}{20}t + C$$

If t=0, P=1000 $\Rightarrow$ $\log 1000 = 0 + C \Rightarrow$ $C = log\ 1000$

Therefore $log\ P = \frac{1}{20}t + log\ 1000$

If t=10 then $log\ P = \frac{1}{20} \times 10 + log\ 1000$, Simplifying we get

$$log\frac{P}{1000} = 0.5 \Rightarrow e^{0.5} = \frac{P}{1000} \Rightarrow 1.648 = \frac{P}{1000}$$

Hence P = ₹1648

2. Form the differential equation of the family of circles touching the y-axis at origin.

Sol: Circle is $(x - a)^2 + y^2 = a^2$

$\Rightarrow x^2 - 2ax + a^2 + y^2 = a^2 x^2 + y^2 - 2ax = 0$ ...(i)

Differentiating w.r.t. x, we get

$2x + 2yy' - 2a = 0 \Rightarrow 2a = 2x + 2yy'$

Substituting in (*i*), we get

$x^2 + y^2 - x(2x + 2yy') = 0$

$\Rightarrow 2xyy' + x^2 - y^2 = 0$ is the required equation.

3. The surface area of a balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 2 seconds its is 5 units. Find the radius after t seconds.

Sol: Let r be the radius of balloon at any time t.

Then $\frac{dA}{dt} = k \quad \Rightarrow dA = k\,dt$ Integrating on both sides we get

$\int dA = k\int dt \Rightarrow A = kt + C \;\Rightarrow 4\pi r^2 = kt + C$

If $t = 0, r = 3$, substituting we get $C = 36\pi$. Hence we get $4\pi r^2 = kt + 36\pi$

If $t = 2, r = 5$ then we get $k = 32\pi$

$$4\pi r^2 = 32\pi\, t + 36\pi \Rightarrow r^2 = 8t + 9 \Rightarrow r = \sqrt{8t + 9}$$

4. Find the general solution of the differential equation $x + y\frac{dy}{dx} = 0$

Sol: Separating the variables x, y we get

$x = -y\frac{dy}{dx} \quad \Rightarrow xdx = -ydy$

Integrating on both sides we get

$$\int xdx = -\int y\,dy \Rightarrow \frac{x^2}{2} = -\frac{y^2}{2} + k$$

$\Rightarrow x^2 + y^2 = 2k$

$\Rightarrow x^2 + y^2 = C$ which is the general solution of the given DE.

And the particular solutions are obtained by changing the arbitrary constant value C.

5. Find the general solution of the differential equation $\log\left(\frac{dy}{dx}\right) = ax + by$

Sol: $\frac{dy}{dx} = e^{ax+by} \Rightarrow \frac{dy}{e^{by}} = e^{ax}\,dx \;\Rightarrow \int e^{-by}\,dy = \int e^{ax}\,dx$

$\Rightarrow -\frac{1}{b}e^{-by} = \frac{1}{a}e^{ax} + C$ which is the required solution.

6. Verify that $y + x + 1 = 0$ is a solution of differential equation $(y - x)\,dy - (y^2 - x^2)\,dx = 0$

Sol: $(y - x)[dy - (y + x)\,dx] = 0 \Rightarrow \frac{dy}{dx} = y + x$ ...(*i*)

Given $y + x + 1 = 0 \Rightarrow \frac{dy}{dx} + 1 = 0 \Rightarrow \frac{dy}{dx} - (y + x) = 0 \Rightarrow \frac{dy}{dx} = x + y$ ...(*ii*)

From (*i*) and (*ii*) get the result.

7. Find the order and degree (if defined) of the differential equation

$$ydx + xlog(\frac{y}{x})dy - 2xdy = 0$$

Sol: On simplifying we get $\left[2x - xlog\left(\frac{y}{x}\right)\right]dy = ydx$

$\Rightarrow \frac{dy}{dx} = \frac{y}{2x - xlog(\frac{y}{x})}$

The highest order derivative present is $\frac{dy}{dx}$and it is raised to power 1. So its order is 1 and degree is also 1.

8. Solve the differential equation xy dy $=$ (y $+$ 5) dx, given that y(5) $=$ 0.

Sol: : $\int \frac{y}{y+5} dy = \int \frac{dx}{x}$

$\int \frac{(y+5)-5}{y+5} dy = \int \frac{dx}{x} \quad \Rightarrow \int \{1 - \frac{5}{y+5}\} dy = \int \frac{dx}{x}$

$\Rightarrow y - 5\, log\, |y + 5| = log\, |x| + C$ ...(*i*)

Given when $x = 5$, $y = 0$ $\quad \Rightarrow 0 - 5\, log\, 5 = log\, 5 + C \Rightarrow C = - 6\, log\, 5$

Substituting in (*i*), we get

$y - 5\, log\, |y + 5| = log\, |x| - 6\, log\, 5$ is the required solution.

9. Verify that $y + x + 1 = 0$ is a solution of differential equation

$$(y - x)\, dy - (y^2 - x^2)\, dx = 0.$$

Sol: $(y - x)[dy - (y + x)\, dx] = 0$

$\Rightarrow \frac{dy}{dx} = y + x$ …....(*i*)

Given $y + x + 1 = 0 \quad \Rightarrow \frac{dy}{dx} + 1 = 0$

$\Rightarrow \frac{dy}{dx} - (y + x) = 0 \Rightarrow \frac{dy}{dx} = x + y$ …...(*ii*)

From (*i*) and (*ii*) we get that $y + x + 1 = 0$ is the solution of the given DE.

10. Find the general solution of $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

Sol: On simplifying and integrating we obtain $\int \frac{1}{1+cos2y}\, dy = \int \frac{1}{1-cos2x}\, dx$

$$\int \sec^2 y\, dy = -\int cosec^2 x\, dx$$

$tany = \cot x + C$ which is the required solution.

EXERCISE QUESTIONS

VERY SHORT ANSWER QUESTIONS (2mark QUESTIONS)

1. Write the sum of the order and degree of the differential equation:

$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + x^4 = 0$ (Ans: 4)

2. Find the differential equation of the family of circles having centre at origin

(Ans: $x + yy' = 0$)

3. Find the general solution of the following differential $x(e^{2y} - 1)dy + (x^2 - 1)e^y dy = 0$

(Ans: $e^y + e^{-y} + \frac{x^2}{2} - \ln|x| = C$)

4. Verify that given function $y = \sqrt{1 + x^2}$ is a solution of the corresponding differential equation $\frac{dy}{dx} = \frac{xy}{1+x^2}$

5. Determine the order and degree (if defined) of the following differential equations

$(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$ (Ans: Order 3, Degree 2)

SHORT ANSWER QUESTIONS (3 mark QUESTIONS)

1. Form the differential equation representing the family of ellipses having foci on the *x*-axis and centre at the origin. (Ans: $xyy' + xy'^2 - yy' = 0$)

2. Find the differential equation representing the family of curves $v = \frac{A}{r} + B$

where *A* and *B* are arbitrary constants (Ans: $r\frac{d^2v}{dr^2}+2\frac{dv}{dr}=0$)

3. Find the equation of the curve passing through the point (1, -1) whose differential equation is

$$xy\frac{dy}{dx}=(x+2)(y+2)$$

4. The amount of oil pumped from one of the wells decreases at the continuous rate of 10% per year. When will the wells output fall to one-fourth of its present value?

5. Half-life of radioactive carbon-14 is 5700 years. A certain bone was observed to contain 75% of carbon-14 as compared to what is present in the leaving creatures. Determine its antiquity.

LONG ANSWER QUESTIONS

1. Find the particular solution of the differential equation $\frac{dy}{dx}$ = x (2logx+1) when x=2, y=0

Solution: $dy=(2\,x\log x+x)dx$

On integrating both sides, we get

$$\int 1\,dy=2\int x\log x\,dx+\frac{x^2}{2}+c$$

$$\Rightarrow y=2\left[\log x\left(\frac{x^2}{2}\right)-\int\frac{1}{x}\frac{x^2}{2}dx\right]+\frac{x^2}{2}+c$$

$$\Rightarrow y=x^2\log x-\frac{x^2}{2}+\frac{x^2}{2}+c$$

$$\Rightarrow y=x^2\log x+c \quad \dots(1)$$

By given x=2, y=0

0 = 4 log 2 +c

$$\Rightarrow c=-4\log 2$$

Substituting, $c=-4\log 2\ in(1)$, weget

$$y=x^2\log x-4\log 2$$

2. In a certain culture of bacteria the rate of increase is proportional to the number present. It is found that there are 10,000 bacteria at the end of 3 hours and 40,000 bacteria at the end of 5 hours. How many bacteria were present in the beginning?

Solution: Let P be the number of bacteria after t hours

$$\frac{dP}{dt}\,\alpha\,P \quad\Rightarrow\quad \frac{dP}{dt}=kP$$

$\Rightarrow \int \frac{dP}{P} = \int kdt \Rightarrow logP = kt + c$

$\Rightarrow P = e^{kt+c} \quad = e^{c}.e^{kt}$

Given P(3)=10,000

P(5)=40,0000

Where $e^{c} = \lambda$

Dividing (3) by (2) , we get $e^{2k} = 4 \quad \Rightarrow e^{2k} = 2$

By substituting$e^{2k} = 2$ *in equation* 2, we get

$\lambda(2)^{3} = 10000 \quad \Rightarrow \lambda = 1250$

From(1) we get,

$P(t) = 1250\, e^{kt}$now,

$P(0) = 1250e^{0} = 1250$

Hence we can say, there were 1250 bacteria in the beginning.

3. a cake is taken out from an oven when its temperature has reached 185°F and is placed on a table in a room whose temperature is 75°F .If the temperature of cake reaches 150 °F after half an hour, what will be its temperature after 45 minutes.

Solution: Let T be the temperature of the cake after t minutes.

By Newton's Law of cooling

$\frac{dT}{dt} = -k(T - 75^{o})$,where k is the constant of proportionality

$\int \frac{dT}{T-75} = -k \int 1dt$

$\log(T - 75) = -kt + c$

$T - 75 = e^{-kt+c}$

$T - 75 = e^{c}e^{-kt}$

$T - 75 = \lambda e^{-kt}$ ... (1)(Wheree^{c}=λ)

Substituting$t = 0,\ T = 185^{0}$in equation(1),we get

$185 - 75 = \lambda \Rightarrow \lambda = 110$

So,$T - 75 = 110\, e^{-kt}$ ...(2)

Substituting,$t = 30,\ T = 150^{0}$in equation(2),we get

$150 - 75 = 110\, e^{-30k}$

$75 = 110e^{-30k}$

$e^{-30k} \quad = \frac{75}{110} = 0.6818 \quad \dots(3)$

$T - 75 = 110\, e^{-45k}$

Substituting,$t = 45$in equation(2),we get

$T = 75 + 110(e^{-30k})^{1.5}$

$T = 75 + 110(0.6818)^{1.5}$(Using(3))

$T = 136.92$ Hence the temperature after 45 minutes is 137°F(approx.)

(LONG ANSWER QUESTIONS)

EXERCISE

1. Solve (x+1) $\frac{dy}{dx}$ =2xy, given that y (2) = 3

2. Find the equation of curve passing through the point (1,-1) whose differential equation is

$$xy\frac{dy}{dx} =(x+2)\ (y+2)$$

3. At any point (x, y)of a curve, the slope of tangent is twice the slope of the line segment joining point of contact to the point (-4,-3). Find the equation of the curve given that it passes through (-2, 1).

Answers

1. log 1y 1 = 2x- 2log 1(x+1)1 + 3 log 3-4

2. y-x +2 = 2 log 1x(y+2)1

3. y+3 = $(x + 4)^2$

<table>
<tr><th colspan="2">CASE BASED QUESTIONS</th></tr>
<tr><td>1.</td><td>

Nembutal, a sodium salt (sodium pentobarbital) acts as a sedative and has many applications. Suppose Nembutal is used to anesthetize a dog. The dog is anesthetized when its blood stream concentration contains at least 45mg of sodium pentobarbital per kg of the dog’s body weight. If the rate of change of sodium pentobarbital say, x in the body, is proportional to the amount of drug present in the body. Show</td></tr>
</table>

that sodium pentobarbital is eliminated exponentially from the dog's blood stream given that its half-life is 5 hours. What single dose should be administered in order to anesthetize a 50 Kg dog for 1 hour?

Solu:

let x be the amount of drug at time

$\frac{dx}{dt} = -kx,$ Where k is the rate at which the drug leaves the blood stream.

$\int \frac{dx}{x} = -k \int 1\, dt$ ➔ $\log x = -kt + c \Rightarrow x = e^{-kt+c}$

$x = \lambda e^{-kt}$, where $\lambda = e^c$

$x = x_0 e^{-kt}$ ... (1) Initial amount of x_0 = drugsince,half-lifeofdrug=5hours

$\therefore \frac{x_0}{2} = x_0 e^{-5k} \Rightarrow e^{-5k} = \frac{1}{2} \Rightarrow e^{5k} = 2$

For a dog that weight 50kg the amount of drug in the body after1hour=

(45mg/kg)x50kg=2250mg

From(1) 2250=$x_0 e^{-k}$(as$t = 1$)

$x_0 = 2250(e^k) = 2250 \text{ x } 2^{1/5} = 2585mg(approx)$

So a single dose of 2585mg should be administered to anesthetizea50kg dog for 1hour.

2. Ms. Rajani deposited Rs. 10000 in a bank that pays 4% interest compounded continuously

a) How much amount will she get after 10 years?

b) How long it will take the money to double?

Solu:

We know , $\frac{dA}{dt} = rA \Rightarrow \int \frac{dA}{A} = \int r\, dt$

$\log A = rt + c \Rightarrow A = e^{rt+c}$

$A = A_o\, e^{rt}$ ... (1)

At $t = 0,$

$A = 10{,}000$

10,000=A_0

Hence A=10,000 $e^{0.04\,t}$ ... (2)

a) $A(10)$=10,000 $e^{0.04 \times 10} = 10,000\ e^{0.4}$(using(2))

$A(10)$ = 10,000x1.49182 = Rs 14918.2

b) We have to find t in which $A_0 = 10,000$ becomes A=20,000.

20,000=10,000$e^{0.04t}$ from (2)

$$e^{0.04t} = 2$$

$$\log e^{0.04t} = log_e 2$$

$$0.04t = log_e 2 \Rightarrow t = \frac{log_e 2}{0.04} = \frac{0.6931}{0.04}$$

$$t = 17.32\ years(approx)$$

3. Rani, student of class XII is very much interested to know about radiocarbon. She gone through many books and came to know that the amount of radiocarbon present after t years is given by $A = A_0 e^{-(\ln 2)\left(\frac{1}{5700}\right)t}$ Where A_0 is the amount present in the living plants and animals.

a) Find the half-life of radiocarbon.

b)Charcoal from an ancient pit contained $\frac{1}{4}$ of the radiocarbon found in living sample of same size. Rani is interested to estimate the age of charcoal

Solu

a) $\frac{A_0}{2} = A_0\ e^{\left(-\frac{ln2}{5700}\right)t}$

$$\left(\frac{-ln2}{5700}\right)t = \ln\left(\frac{1}{2}\right)$$

$$t = \frac{-5700}{ln2}(-ln2) = 5700\ years$$

The half-life is 5700 years.

b) $\frac{1}{4}A_0 = A_0 e^{\left(-\frac{ln2}{5700}\right)t}$

$$-\frac{ln2}{5700}t = ln\left(\frac{1}{4}\right)$$

	$-(\ln 2)t = 5700(\ln 1 - \ln 4)$ $-(\ln 2)t = -2(\ln 2)5700$ The charcoal is about 11,400 year old. $t = 11{,}400$

(CASE BASED QUESTIONS)

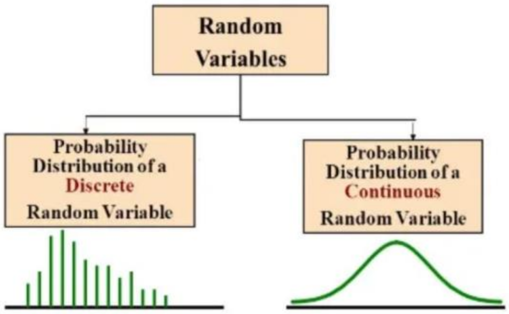
EXCERCISE

1.	A rumor on WhatsApp spreads in a population in 5000 people at a rate proportional to the product of the number of people who have heard it and the number of people who have not. Also, it is given the 100 people initiate the rumor and a total of 500 people know the rumor after 2 days.  Based on the above information, answer the following questions (a) If y(t) denote the number of people who know the rumour at an instant t, then what is the maximize value of y(t). (b)What will be the value of y (2)? (c)What will be the value of y at anytime instant t?
2.	Shalini, 15 year old girl was playing with a spherical balloon and observed that the volume of balloon being inflated changes at a constant rate. Also, she observed that initial radius of balloon is 3 units and after 3 seconds it is 6 units.  (a) Find radius of balloon after T seconds. (b)Find the volume of balloon after t seconds

Answers: (1) (a) 5000 (b) 500 (c) $\frac{5000}{49e^{-5000kt}+1}$

(2) (a) r $=[9(7t+3)]^{\frac{1}{3}}$ (b) 12 π (7t+3) cu.unit

UNIT 4: PROBABILITY DISTRIBUTIONS

CONCEPT	DEFINITION
Definition and example of discrete and continuous random variable and their Distribution. Different Probability Distributions  Probability Distribution of Discrete and Continuous Random Variable.	A discrete distribution is one in which the data can only take on certain values. For Example, integers. A continuous distribution is one in which data can take on any value within a specified range (which may be infinite).
The expected value (Mean) of discrete random variable as summation of product of discrete random variable by the probability of its occurrence.	The expected value, E(X), or mean μ of a discrete random variable X, simply multiply each value of the random variable by its probability and add the products. The formula is given as $E(X)= \text{Mean}=\mu=\sum x\, P(x)$
Variance	$\text{Var}(X) = E(X^2) - [E(X)]^2$ Variance is a measure of dispersion, meaning it is a measure of how far a set of numbers is spread out from their average value.

Standard deviation (Characteristics of the binomial distribution: Identify the Bernoulli Trials and apply Binomial Distribution)	Standard deviation is square root of the variance. $\sigma_X{}^2 = \mathrm{Var}[X]$ A standard deviation (or σ) is a measure of how dispersed the data is in relation to the mean. Low, or small, standard deviation indicates data are clustered tightly around the mean, and high, or large, standard deviation indicates data are more spread out.
Binomial Distribution: (The binomial distribution is the discrete probability distribution that gives only two possible results in an experiment, either Success or Failure).	**Binomial formula:** $P(r) = n_{Cr}\, p^r q^{n-r}$ Where n = number of trials P = probability of success q = probability of failure Mean =np Variance = npq Standard Deviation = $\sqrt{npq}$
Poison Distribution :	**Poisson formula:** $P(x) = \frac{\lambda^X e^{-\lambda}}{X!}$ Mean = Variance = λ A Poisson distribution is a discrete probability distribution. It gives the probability of an event happening a certain number of times (k) within a given interval of time or space. The Poisson distribution has only one parameter, λ (lambda), which is the mean number of events.

Normal Distribution:

Definition:

A probability distribution that is symmetric about the mean, showing that data near the mean are more frequent in occurrence that data far from the mean. Normal distribution is also kwon as the gaussian distribution.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

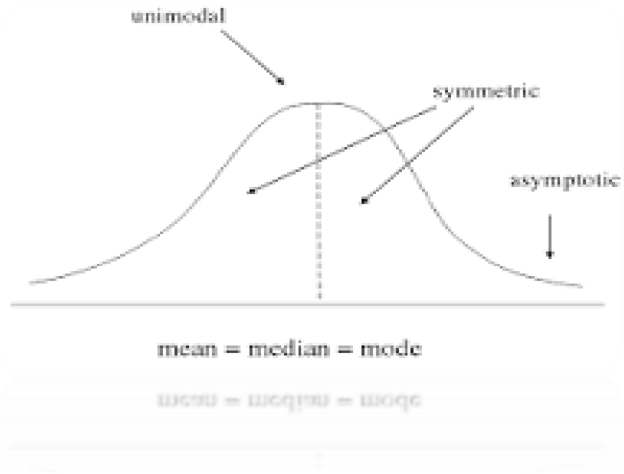
$f(x)$ = probability density function

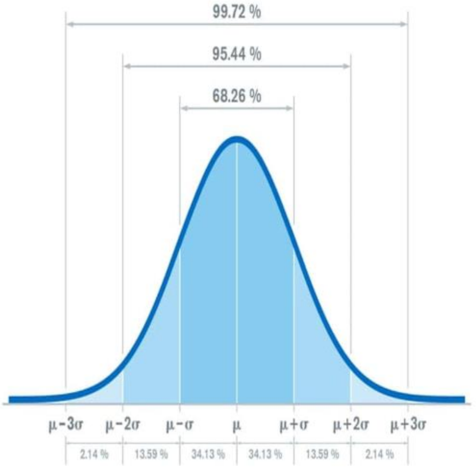
σ = standard deviation

μ = mean

Formula:

Standard Normal Deviation:





Definition:

A standard Normal distribution is a normal distribution with a mean 0 (zero) and a standard deviation is 1(one).

$Z = \frac{x-\mu}{\sigma}$ where

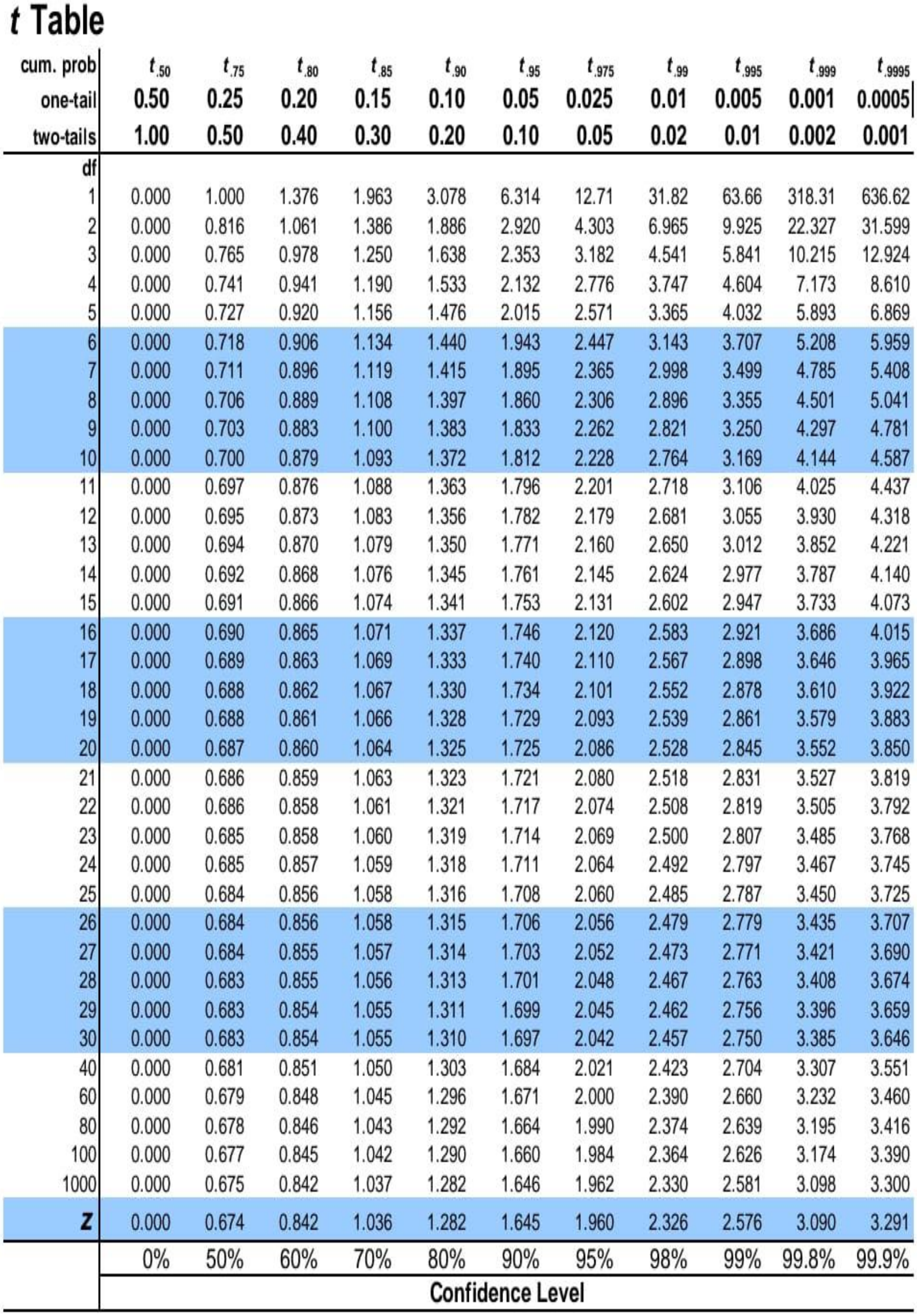
x = value of the random variable

μ = mean

σ = S.D.

Properties:

- The total area under normal curve is 1
- A normal distribution comes with a perfectly symmetrical shape.
- There is no skewness in standard normal distribution
- No portion of curve lies below x-axis so probability can never be negative.
- The curve becomes parallel to x-axis which is supposed to meet it at infinity.



t Table

cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
two-tails	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										

ASSERTION – REASON BASED QUESTIONS

In the following questions, a statement of **assertion (A)** is followed by a statement of **Reason (R).** Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true and R is the not the correct explanation of A.

(c) A is true but R is false.

(d) A is false but R is true.

<table>
<tr><td>Q.1</td><td>**Assertion(A):** Probability Distribution When a coin tossed twice is considered a continuous Distribution

Reason (R): A continuous distribution is one in which data can take on any value within a specified range (which may be infinite).

Solution: By definition,Out comes are integers which is not a continuous interval.

Answer: option (d)</td></tr>
<tr><td>Q.2</td><td>**Assertion(A):** If X is a random variable with the following distribution
<table>
<tr><td>X</td><td>A</td><td>B</td></tr>
<tr><td>P(X)</td><td>P</td><td>q</td></tr>
</table>
Then the relation between p and q is p+q=1

Reason (R): If X: S → $R$is a discrete random variable with range {p,q}then p+q=1.

Solution: By definition of Probability distribution, the sum of P(x) for x$\epsilon\ S\ is$ 1.
Answer: option (a)</td></tr>
<tr><td>Q.3</td><td>**Assertion(A):** if 8 coins are tossed simultaneously then probability of getting 1 head is $8(\frac{1}{2})^8$.
Reason (R): The binomial distribution is the discrete probability distribution that gives only two possible results in an experiment, either Success or Failure.

P(r) = n_{Cr} p^r q^{n-r}Where n = number of trials

P = probability of successq = probability of failure

Answer: option (a)</td></tr>
<tr><td>Q.4</td><td>**Assertion(A):** The mean and variance of a binomial variate are 2.4 and 1.44 Respectively,then q=$\frac{2}{5}$</td></tr>
</table>

	Reason (R): If X∼ B(n,p) then mean and variance of X are np and npq respectively. **Solution:** Mean=μ= np= 2.4 …………...(i) $and\ variance = \ \sigma^2 = 1.44 \Longrightarrow npq = 1.44$……………. (ii) Solving (i) and (ii),we get q=3/5 **Answer:** option (d)
Q.5	**Assertion(A):** When a coin is tossed n times if the probability for getting 6 heads is equal to the probability of getting 8 heads, then n=14. **Reason (R): P(r) = n_{Cr} p^r q^{n-r} ,Where n = number of trials ,P = probability of success ,q = probability of failure** **Solution: Given P(x=6)=P(x=8)** **⟹ n_{C6} p6 q^{n-6}= n_{C8} p8 q^{n-8} ,Where p=q=$\frac{1}{2}$** **On solving we get n=14.** **Answer:** Option (a)
Q.6	**Assertion(A):** For a Poisson variate X,P(x=2)=P(x=3),then variance of X is 3. **Reason (R):** Poisson formula: P(x) = $\frac{\lambda^X e^{-\lambda}}{X!}$; Where Mean = Variance = λ **Solution:** P (x= 2) =P(x=3)⟹ $\frac{\lambda^2 e^{-\lambda}}{2!}=\frac{\lambda^3 e^{-\lambda}}{3!}$ ⟹ On solving we get λ= 3. **Answer:** option (a)
Q.7	**Assertion(A):**If the mean of a Poisson distribution is 2.56 then standard deviation is 1.6 **Reason (R):** The Poisson distribution has only one parameter, λ (lambda), which is the mean number of events.In Poisson distribution is Mean = Variance = λ. **Solution:**We know Variance=Mean=λ=2.56 Then Standard deviation =$\sqrt{variance}$ =$\sqrt{2.56}$=1.6 **Answer:** option (a)
Q.8	**Assertion(A):** Suppose a fair coin is tossed 2 times as head is out come then range of X={0,1,2} **Reason (R):** If X:S→ R is a random variable and k∈R then X(k)= {a∈ S where X(a)= K} **Solution:** When outcomeis TT, x=0 When outcomesareTH, HT, If x=1 When outcome is HH, x=2 **Answer**: option (a)

Q.9

Assertion(A): If X is a Poisson random variable with Parameter 3,then mean of X is 3 and variance of X is also 3.

Reason (R): If X is a Poisson random variable with Parameter λ then mean of X is λ and variance of X is λ^2.

Solution: In Poisson distribution, we know
mean = variance.

Answer: Option (C)

Q.10

In a binomial distribution n=200,p=0.04.Taking Poisson distribution is an approximation to the binomial distribution .

Assertion(A): Mean of the Poison distribution =8

Reason (R): In a Poisson distribution,P(X=4)=$\frac{512}{3e^8}$

Solution : Meanof Poisson ditribution =np=(200)(0.04)=8 and P(X=4) = $\frac{e^{-8}8^4}{4!}$ =$\frac{512}{3e^8}$

Answer: Option (b)

MULTIPLE CHOICE QUESTIONS

Q.1

Describe sample space associated with flipping a coin until either heads or tails occur twice from the following options.

(a) {HH,THH,HTHT,TT,HTT,THT}
(b) {HH,THH,HTH,TT,TTH,THT}
(c) {HH,HHT,HTH,TT,TTH,THT}
(d) {HHT,THH,HTH,TT,HTT,THT}

Answer :option(b)

Q.2

A discrete random variable X has the probability distribution given as below

X	0.5	1	1.5	2
P(X)	K	K^2	$2K^2$	K

Find the value of k?

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{3}$

Solution: since in a random experiment sum of Probabilities of outcomes in a sample space , is equal to one.

Answer: option (d)

Q.3	Let X be a Discrete random variable assuming values $x_1, x_2, x_3,......x_n$ with probabilities p_1, p_2 , $p_3….p_n$ respectively. Then variance of X is given by (a) $E(X^2)$ (b) $E(X^2)+E(X)$ (c) $E(X^2)-E(X)$ (d) $\sqrt{E(x^2)-E(X)}$ **Solution:** By the definition of variance Correct option is (c)
Q.4	In a Poisson Distribution, if ‘n’ is the number of trials and ‘p’ is the probability of success, then the mean value is given by? a) m = np b) m = $(np)^2$ c) m = np(1-p) d) m = p **Answer: (a)** **Explanation:** (For a discrete probability function, the mean value or the expected value is given byMean(μ)=∑xp(x) For Poisson Distribution P(x) = $\frac{\lambda^X e^{-\lambda}}{X!}$, substitute in above equation and solve to get μ = m = np).
Q.5	The probability that a student is not a swimmer is $\frac{1}{5}$.then the probability that out of five students four are swimmers is (a) $5_{C4}\,(\frac{4}{5})^4(\frac{1}{5})$ (b) $(\frac{4}{5})^4(\frac{1}{5})$ (c) $5_{C0}\,(\frac{4}{5})^4(\frac{1}{5})$ (d) none of these **Answer:** Option (a)
Q.6	The shape of the Normal Curve is a) Bell Shaped b) Flat c) Circular d) Spiked **Answer:** (a) **Explanation:** Due to the nature of the Probability Mass function, a bell-shaped curve is obtained.
Q.7	For a standard normal variate, the value of mean is? a) ∞ b) 1 c) 0 d) not defined **Answer:** (c) **Explanation:** For a normal variate, if its mean = 0 and standard deviation = 1, then It’s called as Standard Normal Variate. Here, the converse is asked.
Q.8	Skewness of Normal distribution is a) Negative b) Positive c) 0 d) Undefined

	Answer: (c) **Explanation:** Since the normal curve is symmetric about its mean, its skewness is zero.
Q.9	**In the manufacture of glassware, bubbles can occur in the glass which reduces the status of the glassware to that of a second. If on average one in every 1000items produced has a bubble. Calculate the probability of three thousand are seconds** (a) 1 (b) 0.5 (c) -0.5 (d)0.4\\ **Solution:** suppose that X= number of items with bubbles then $X\sim B$ (3000,0.001) Since n=3000> 100 and p=0.001<0.005 we can use the Poisson Distribution withλ=np=(3000)(0.0001)=3.The calculation is P(X=6)≈0.05 **Answer:** Option (b)
Q.10	Most graduate schools of business require applicants for admission to make the graduate management admission to take the graduate management admission council's GMAT examination. Scores on the GMAT are roughly normally distributed with a mean of 527 and a standard deviation of 112.what is the probability of an individual scoring above 500 on the GMAT? (a)0.5 (b)0.59 (c)0.6 (d)1 **Solution:** Normal distribution: Z=$\frac{x-\mu}{\sigma}=\frac{500-527}{112}$=-0.24107

ASSERTION – REASON PRACTICE QUSTIONS

1	**Assertion**: In Poisson Distribution mean and variance are equal. **Reason:** In a Poisson Distribution, Mean = m and Standard Deviation = $(m)^{1/2}$ **Answer:** option (a)
2	**Assertion:** Variance of the data 2,4,5,6,8,17,is 23.33 then variance of 4,8,10,12,16,34 will be 46.66. **Reason:** Variance = npq; Where n = number of trials P = probability of success q = probability of failure **Answer:** option (b)
3	**Assertion:** If X is a Poisson random variate with mean 2,then P$\|X-2)\|$<1 will be 2(e^{-2}). **Reason:** In a Poisson distribution X assigns only integer values.

Answer: option (a)

4 **Assertion(A):** If X is a random variable with the following distribution

X	0	1	other
P(X)	k	2k	0

Then value of k=$\frac{1}{3}$.

Reason (R): If X: S $\rightarrow R$is a discrete random variable with range {p, q} then sum of probabilities is equal to 1.

Solution:

By definition of Probability distribution, the sum of P(x) for x$\epsilon\ S\ is$ 1.

Answer: option (a)

5 The length of the item produced is normally distributed with mean 5.3cm and standard deviation is 2.5 cm

Assertion: probability the randomly selected item is with length less than 4.5cm is 37.45%

Reason: According to normal Distribution Z=$\frac{x-\mu}{\sigma}$,where x-value of random variable ,

$\sigma\ -standard\ deviation\ ,\mu$- Mean .

Answer: Option (a)

PRACTICE QUESTIONS (M.C.Q)

1 **A die is thrown 10 times, the probability that an odd number will come up at least one time is**

a. 1013/1024 b. 1/1024 c. 1023/1024 d. 11/1024

Answer: Option (c)

2 The probability of guessing correctly at least 8 out of 10 answers on a true-false type examination is

a.$\frac{7}{64}$ b.$\frac{7}{128}$ c. $\frac{45}{1024}$ d.$\frac{7}{41}$

Answer: option (b)

3 Eight coins are tossed together then probability of getting exactly 3 heads is

a. $\frac{1}{256}$ b. $\frac{7}{32}$ c. $\frac{5}{32}$ d. $\frac{3}{32}$

Answer: option (c)

4 Which one is not a requirement of a binomial distribution.

a. There are 2 outcomes for each trial.
b. The probability of success must be same for all trails
c. There is a finite number of trails
d. The outcomes must be dependent on each other

Answer: option (d)

5 The time taken to assemble a Mobile in a manufacturing unit is a random variable having a normal distribution of 20 hours and a standard deviation of 2 hours. What is the probability that a mobile can be assembled at this manufacturing unit in a period of time Between 20 and 22 hours

a.0.3413 b. 0.3414 c. 0.3412 d. 0.4013

Answer: option (a)

VERY SHORT ANSWER QUESTIONS

1 A die is rolled. If a random variable X is defined as the number on the upper

face , then find its probability distribution.

Solution:

Let sample space S = { 1,2,3,4,5,6 }

Probability distribution :

X	1	2	3	4	5	6
P(X)	1/6	1/6	1/6	1/6	1/6	1/6

2 A game at a school fair costs Rs 10 with a prize of RS 50.If the probability of winning is 1/10. Is it a fair game ?

Solution:

X	50 (winning)	0(losing)
P(X)	1/10	9/10

Mean =$\sum x_i P(x_i)$ = $50 \times \frac{1}{10}$ + 0× $\frac{9}{10}$ =50/10 = Rs 5 (GAIN)

Cost of game is Rs 10. NOT A FAIR GAME

3 In a certain village some families are strictly limited to two children. The probability distribution of number of children is given below. Find the mean number of children.

No of children	0	1	2
Probabilty	1/10	1/2	2/5

Solution:

X	0	1	2
P(X)	1/10	1/2	2/5

Mean =$\sum x_i P(x_i)$ ==$\left(0 \times \frac{1}{10} + 1 \times \frac{1}{2} + 2 \times \frac{2}{5}\right)$

= ½ + 4/5 = $\frac{(5+8)}{10}$ = 13/10 =1.3

4 If X follows binomial distribution with parameters n=5, P(X =2) =9 P(X=3).

Find the value of p.

Solution:

Given : n =5 and P(X =2) =9 P(X=3).

∴ $5C_2 p^2 q^3$ = 9 $5C_3 p^3 q^2$

q = 9 p , p + q =1 , 10p=1

p =1/10.

5 For the Poison Distribution , Find P(2), given $\lambda = 0.7$

Solution:

P (2) = $\frac{(\lambda^2)(e^{-\lambda})}{2!}$ = $\left(\frac{0.7^2}{2}\right) \times e^{-0.7}$ = $\frac{0.49 \times 0.497}{2}$ = 0.1218

6 If X is a normal distribution of a random variable with mean 12 and standard deviation 3, Find P (X< 15).

Solution:

Given μ = 12 and σ ==3

	Z = $\frac{x-12}{3}$,, For x =15, Z = $\frac{15-12}{3}$ = 3/3 = 1 P(X< 15) = P (Z = 1) = F (1) =0.8413.
7	A Fair coin is tossed 6 times. Find the probability of getting head atleast 4 times ? **Solution:** Coin tossed 4 times. n = 4, p =1/2 and q = 1 – ½ = ½ P (Atleast head 4 times) = P (X =4) + P (X=5) + P(X=6) $\left(6C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2\right)+\left(6C_5\left(\frac{1}{2}\right)^5\frac{1}{2}^1\right)+\left(6C_6\left(\frac{1}{2}\right)^6\right)$ $\left(\frac{1}{2}\right)^6(15+6+1)$ = $\frac{22}{64}$ $=\frac{11}{32}$.
8	A Radar unit is installed to measure the speed of cars in highway. The speeds are normally distributed with mean 80 km/hr and standard deviation 10 km/hr. Find the probability of a car running at lessthan 60 km/hr? **Solution:** Given : $\lambda = 80\frac{km}{hr}\,and\ \ \sigma = 10\frac{km}{hr}$. P(X < 60) = P (Z <$\frac{60-80}{10}$) = P (Z < -2) F (-2) = 1 – F (2) = 1 -0.9772 = 0.0228
9	If 2 % of books bound in a certain workshop have a defective binding. Find the probability that if 5 books out of 400 books will have defective binding ? **Solution:** Given : p = 2% = 2/100 and λ = np = 400 × $\frac{2}{100}$ = 8 P (X =5) = $\frac{\lambda^5 e^{-\lambda}}{5!}$ = $\frac{8^5 e^{-8}}{120}$ = $\frac{32768\ \times 0.00034}{120}$ = 0.0928
10	In a Binomial Distribution, if mean is 5and variance is 4, Find the number of trials ? **Solution:**

$\frac{Variance}{Mean} = \frac{npq}{np} = q = \frac{4}{5}$, $p = 1 - q = \frac{1}{5}$

Also n p =5 , number of trials n =25

11 Four bad oranges are mixed with 16 good oranges. Find the probability distribution

Number of bad oranges in a draw of two oranges.

Answer:

x	0	1	2
P(x)	60/95	32/95	3/95

12 A boy throws a coin. He is to get two rupees for getting a head. Find his expectation?

Answer : 6

13 Find the mean of the Binomial Distribution B (8 ,1/4)

Answer : 2

14 A die is thrown twice and success is getting an odd number. Find the mean and variance of success.

Answer : Mean =1 and Variance = 1/2

15 Suppose a book of 614pages contains 43 typograprical errors and if 10 pages are selected randomly, what is the probability that they are free of errors ?

Answer : P(x =0) =0.497

SHORT ANSWER QUESTIONS

1 A Random variable x has the following probability distribution, where k is constant

x	0	1	2	otherwise
P(x)	k	2k	3k	0

(i) Determine the value of k ?

(ii) Find P (x < 2), P(x$\leq$ 2), $P(x \geq 2)$

Solution:

(i) P (X =0) + P (X=1) + P (X =2) + P (X > 2) =1

k + 2k +3k + 0 =1 , 6k =1, k=1/6

(ii) P (X < 2) = P(X =0) + P (X =1) = k + 2k =3k =3/6=1/2

P (X≤ 2) = k + 2k + 3k =6k = 1

P (X≥ 2) = 3k + 0 =3k =1/2

2 Two cards are drawn simultaneously from a well shuffled pack of 52 cards . Find the probability distribution of number of jacks.

Solution:

Total no of cards :52 (4 jacks + 48 non jacks)

x	0	1	2
P(x)	$\left(\frac{48C_2}{52C_2}\right) = \frac{188}{221}$	$\left(\frac{4C_1 \times 48C_1}{52C_2}\right) = \frac{32}{221}$	$\left(\frac{4C_2}{52C_2}\right) = \frac{1}{221}$

$\sum P(X) = \frac{(188+32+1)}{221}$ =221/221 = 1

3 Find the Mean and Variance of the following distribution.

x	0	1	2	3
P(x)	1/6	1/2	3/10	1/30

Solution:

	p_i	$x_i p_i$	$\left(p_i({x_i}^2)\right)$
0	1/6	0	0
1	½	1/2	½ =5/10
2	3/10	6/10	12/10
3	1/30	1/10	9/30 =3/10
		12/10 =6/5	$\frac{(0+5+12+3)}{10}$ =20/10=2

Mean(μ) = $\sum x_i p_i$ = 6/5 =1.2

Variance = $\sum\left(p_i(x_i{}^2)\right)$ - μ^2 = 2 – 1.44 = 0.56

4 A Random variable x has the following probability distribution,where k is constant

x	0.5	1	1.5	2.5
P(x)	k	k^2	$2k^2$	k

(i) Determine the value of k ?

(ii) Find Mean and Variance of the distribution.

Solution:

x_i	p_i	$x_i p_i$	$\left(p_i(x_i{}^2)\right)$
0.5	k	0.5k	1/6
1	k^2	k^2	1/9
1.5	$2k^2$	$3k^2$	3/9
2	k	2k	4/3
		$4k^2 + 2.5k$	9/6 + 4/9 =105/54

(i)

$\sum p_i$ =1, 2k + 3 k^2 =1, $3k^2 + 2k$ -1 =0

(3k – 1) (k +1) =0, since k > 0, 3k-1=0, k = 1/3

(ii)Mean(μ) = $\sum x_i p_i = 4\left(\frac{1^2}{3}\right) + \frac{5}{2}\times\frac{1}{3}$ = 4/9 + 5/6 =$\frac{(24+45)}{54}$ = 69/54

Mean = $\frac{23}{18}$

Variance = $\frac{105}{54}$ - $\frac{529}{324}$ = $\frac{630-529}{324}$ = $\frac{101}{324}$

5 A die is thrown 5 times. Find the probability of getting an odd number

(i) Atleast 4 times (ii) maximum 3 times

Solution:

Probability of head = p =1/2 and q = 1-1/2 =1/2

n=5

	p (r) = n C_r p ^{r} q $^{n-r}$ = 5 $C_r\left(\frac{1}{2}\right)^{r+5-r}$ = 5 $C_r\frac{1}{32}$ (i) p (r =4) + p (r =5) = $(5C_4 + 5C_5)\frac{1}{32}$ = $\frac{5+1}{32}$ = 6/32 = 3/16 (ii) p (r=0) + p (r=1) + p (r =2) + p (r=3) = 1 -[p (r=4) + p (r =5)] 1 – 3 /16 = $\frac{16-3}{16}$ = $\frac{13}{16}$.
6	Three cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the mean and variance of number of red cards drawn. **Solution:** Probability of getting a red card = 26/52 =1/2 P=1/2 and q = 1 - ½ = ½ And n =3 Mean = np = 3/2 and Vaiance = npq = ¾
7	If the binomial distribution X follows , mean 3 and Variance 3/2, Find P(X $\leq$ 5) **Solution:** $\frac{Variance}{Mean}$ = $\frac{npq}{np}$ = $\frac{\frac{3}{2}}{3}$, q = ½ and p =1/2 Mean = np =3, n =3/p =6 P (X $\leq$ 5) = 1 - P (X=6) 1 - $[6\,C_6\ p^6q^0]$ = 1 - $\frac{1}{2^6}$ = 1 - $\frac{1}{64}$ = $\frac{63}{64}$
8	If the sum of mean and variance of the binomial distribution of 54 trials is 30. Find the Binomial distribution. **Solution:** Given np + npq = 30 (1 + q) np =30 ∴ (1 + q) 54 (1 –q) =30 (1 - q ^{2}) = $\frac{30}{54}$, 1 - $\frac{30}{54}$ = q ^{2} , q ^{2} = $\frac{24}{54}$ = $\frac{4}{9}$ $\therefore\ q = \frac{2}{3}\ \ and\ \ p = 1/3$

	Binomial distribution is (p + q) ^{n} = $\left(\frac{1}{3}+\frac{2}{3}\right)^{54}$
9	Experiance shows that 1.4 % of telephone calls received are wrong numbers. Determine the probability that among 150 calls received 2 are wrong numbers. **Solution:** Given : n =150 and p = 1.4 % = 14 /1000 $\therefore\ \lambda = np\ = 150\ \times \frac{14}{1000}$ = 2.1 P(X =2) = $\frac{(\lambda^2 e^{-\lambda})}{2!}$ = $\frac{4.41\ \times 0.122}{2}$ = 4.41 × 0.061 = 0.269
10	Assume that the probability that a bomb dropped from an aeroplane will hit a target is 1/5. If 6 bombs are dropped , Find the probability that (i) Atleast two will hit the target ? (ii) Exactly 2 will hit the target? **Solution:** Given : n=6 , p= 1/5 λ = 6/5 =1.2 (i) P (X ≥ 2) = 1 - [P (X =0) + P (X=1)] =1 – [$e^{-\lambda} + \lambda e^{-\lambda}$] = 1 - $e^{-\lambda}(1+\lambda)$ = 1 - $e^{-1.2}$ (1 +1.2) =1 – (0.301× 2.2) = 1 – 0.6622= 0. 3378. (ii) P (X =2) =$\frac{\lambda^2 \times e^{-\lambda}}{2!}$ $=$ $\frac{1.44\ \times 0.301}{2}$ = 0.2167
11	For a certain type of laptops the charging time of batteries is normally distributed with mean 50 hours and standard deviation 15 hours. Arun has one of these laptops, Find the probability that the charging time of battery will be between 50 to 70 hours . **Solution:** Given : μ= 50 and σ =15 P (50 < x < 70) = P $\left(\frac{50-50}{15} < Z\ < \frac{70-50}{15}\right)$ P $(0\ < Z < 1.33)$ = F (1.33) – F (0) = 0.9082 - 0.5 = 0.4082
12	For a Poison distribution, 3 P (X =2) = P (X =4). Find P (X =3) . Given e^{-6} = 0.00248. **Solution:**

Given : 3 P (X=2) = P (X=4)

$3\left(\frac{\lambda^2 e^{-\lambda}}{2!}\right) = \left(\frac{\lambda^4 e^{-\lambda}}{4!}\right)$

$\therefore \lambda^2 = 36$, $\lambda = 6$

P (X =3) $= \left(\frac{6^3 e^{-6}}{3!}\right) = \frac{216 \times 0.00248}{6}$ = 0,08928

LONG ANSWER QUESTIONS

1 If 1% of of the electric bulbs manufactured by a company are defective, find the probability that in a sample of 100 bulbs, the number of defective bulbs will be 0,1,2,3,4,5 respectively. Use recurrence relation of poisson distribution. Also find the probability that (i) three or more (ii) less than or equal to 2 bulbs will be defective

Solution :

P=1%=0.01
N=100
λ=np=0.01x100=1
P(0)=$e^{-\lambda} = e^{-1}$= 0.368
Recurring relation of Poisson distribution is
P(r+1)= $\frac{\lambda}{r+1}$ P(r)
P(1)= $\frac{\lambda}{1}$ P(0) = 1x 0.368= 0.368
P(2)= $\frac{\lambda}{2}$ P(1)= $\frac{1}{2}$ x 0.368 =0.184
P(3) = $\frac{\lambda}{3}$ P(2)= $\frac{1}{3}$ x 0..184 =0.0613
P(4)= $\frac{\lambda}{4}$ P(3)= $\frac{1}{4}$ x 0.0613 =0.0153
P(5)= $\frac{\lambda}{5}$ P(4)= $\frac{1}{5}$ x 0.0153 =0.0031

(i) P(X$\geq$ 3) =1-{ P(0)+P(1)+P(2)}

= 1-(0.368+0.368+0.184)
=1-0.92
=0.08

(ii)P(X$\leq$ 2) =P(0)+P(1)+P(2)

=0.368+0.368+0.184
=0.92

2

Find the Probability distribution of the number of Successes of two tosses of a die. Where a Success is defined as “the number greater than 4”. Also find the Mean. Variance and Standard deviation of the distribution.

Solution:

When a die is tossed S={1.2.3.4.5.6}

Let E be the event ‘’the number greater than 4’’

E={5.6}

n (E)=2

p=P(E) =$\frac{2}{6} = \frac{1}{3}$

q=1- $\frac{1}{3} = \frac{2}{3}$

As the die is tossed twice there are 2 bernoullian trials.

Let X denote the number of successes the X can take values 0,1,2

P(0) = 2C$_0$ $p^0q^2 = (\frac{2}{3})^2 = \frac{4}{9}$

P(1) = 2C$_1$ $p^1q^1 = 2\left(\frac{1}{3}\right)(\frac{2}{3}) = \frac{4}{9}$

P(2) = 2C $p^2q^0 = (\frac{1}{3})^2 = \frac{1}{9}$

Probability distribution of Number of successes is

x	0	1	2
P(x)	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Mean =$\Sigma x_i p_i$=(0)$(\frac{4}{9}) + (1)\left(\frac{4}{9}\right) + 2\left(\frac{1}{9}\right) = \frac{6}{9}$

$\Sigma p_i x^2$ = $(0)(\frac{4}{9}) + (1)\left(\frac{4}{9}\right) + 4\left(\frac{1}{9}\right) = \frac{8}{9}$

Variance= $\Sigma p_i x^2$ – $(\Sigma x_i p_i)^2$ =$\frac{8}{9} - \frac{4}{9} = \frac{4}{9}$

Standard Deviation= $\sqrt{Var}$ =$\frac{2}{3}$.

3

Entry to a certain University is determined by a national test. The scores on this test are normally distributed with a mean of 500 and a standard deviation of 100. Tom wants to be admitted to this university and he knows that he must score better than at least 70% of the students who took the test. Tom takes the test and scores 585. Tom does better than what percentage of students?

Solution:

	Given, $\mu=500$ (mean) $\sigma=100$ (standard deviation) Tom does better than what percentage? Tom got 585 marks we will have to find how many students got less than 585 $x<585$ (find the probability) converting the problem in standard form $Z=(x-\mu)/\sigma$ for $x=585$ $Z=(585-500)/100=0.85$ $P(Z<0.85)=P(Z<0)+P(0<Z<0.85)$ $P(Z<0)=0.5$ $P(0<Z<0.85)=0.3023$ (from Z- table) so the total probability$=0.5+0.3023=0.8023$ so Tom got more than 80.23% of students
4	If X is normally distributed with mean 6 and standard deviation 5, find: (i) $P[0\leq X\leq 8]$ (ii) $P(\vert X-6\vert<10)$ Here $P(0<Z<1.2)=0.3849$ $P(0<Z<0.4)=0.1554$ $P(0<Z<2)=0.4772$ **Solution :** Given $\mu=6,\sigma=5$ (i) $P(0\leq X\leq 8)$ We know that $Z=(X-\mu)/\ \sigma$ When $X=0$,$Z=(0-6)/5)/\ =-6/5=-1.2$ When $X=8$, $Z=(8-6)/5=2/5=0.4$ $\therefore P(0\leq X\leq 8)=P(-1.2<Z<0.4)$ $=P(0<Z<1.2)+P(0<Z<4)$

(due to symmetry)

=0.3849+0.1554=0.5403

(ii) P(|X−6|<10)=P(−10<(X−6)<10)

⇒P(−4<X<16)

When X=−4,Z=(−4−6)/5=−10/5=−2

When X=16, Z=(16−6)/5)=10/5=2

P(−4<X<16)

=P(−2<Z<2)=2P(0<Z<2)

=2(0.4772)=0.9544

CASE STUDY QUESTIONS

1 Read the following passage and answer the questions given below (Internal Choice is in option iii.) (Mark 1 + 1 + 2)

Let X denote the number of hours a person watches television during a randomly selected day. The probability that X can take the values x, has the following form, where ‘k’ is some unknown constant.

$$P(X = x) = \begin{cases} 0.2 & if\ x = 0 \\ kx & if\ x = 1\ or\ 2 \\ k(5 - x) & if\ x = 3 \\ 0 & otherwise \end{cases}$$

(i) Find the value of k

(ii) What is the Probability that the Person watches 2 hrs of Television on a selected day

(iii) What is the Probability that the Person watches Television at least 2 hrs on a selected day

OR

What is the Probability that the Person watches Television at most 2 hrs on a selected day

x	0	1	2	3	4
P(X=x)	0.2	k	2k	2k	0

Solution:

(i)Since $\sum P = 1 \Longrightarrow 0.2 + k + 2k + 2k = 1$
$\Longrightarrow 0.2 + 5k = 1 \Longrightarrow 5k = 0.8 \Longrightarrow$
k $= \frac{0.8}{5} = \frac{4}{25}$

(ii) P(X=2) = 2k = $\frac{8}{25}$
(iii)P$(X \geq 2) = 4k = \frac{16}{25}$

OR

$P(X \leq 2) = 0.2+3k = 0.2+\frac{12}{25} = \frac{17}{25}$

2 A factory produces bulbs, of which 6% are defective bulbs in a large bulk of bulbs.

Based on the above information, answer the following questions

(i) Find the probability that in a sample of 100 bulbs selected at random none of the bulbs are defective (Use $e^{-6} = 0.0024$)
(ii) Find the probability that the sample of 100 bulbs has exactly two defective bulbs.
(iii) Find the probability that the sample of 100 bulbs will include not more than one defective bulb.

OR

Find the Mean and Variance of the distribution of number of defective bulbs in a sample of 100 bulbs

	Solutions N=100, p=6/100, λ=np =100x(6/100) =6 P (r)=$\frac{e^{-\lambda}\lambda^{r}}{r!}$ (i) P(0) =$\frac{e^{-6}6^{0}}{0!}$ = e^{-6} =0.0024 (ii) P(2)= $\frac{e^{-6}6^{2}}{2!}$ = 0.0024x ($\frac{36}{2}$) = 0.0432 (iii) P(0)+P(1)= $e^{-6}+6e^{-6}=7e^{-6}$=0.0168 OR Mean = λ =6 Variance=λ=6

QUESTIONS FOR PRACTICE

LONG ANSWER QUESTIONS

1	The I.Q.'s of army volunteers in a given year are normally distributed with mean μ =110 and standard deviation (σ) = 10. The army wants to give advanced training to 20% of those recruits with the highest scores. What is the lowest I.Q. score acceptable for the advanced training? (Ans 118.4)
2	It is known that 3% of plastic buckets manufactured in a factory are defective. Using the Poisson distribution on a sample of 100 buckets, find the probability of: (i) Zero defective buckets (ii) At most one bucket is defective [Use e −3 = 0.049] (Ans (i) 0.049 (ii) 0.196)

CASE STUDY QUESTIONS

1	The Probability distribution of a random variable X is given by $p(X=x)=\begin{cases} kx^2 & if\ x=1,2,3 \\ 2kx & if\ x=4,5,6 \\ 0 & otherwise \end{cases}$ where k is a constant. $Based\ on\ the\ above\ information\ answer\ the\ following.$ (i) $What\ is\ the\ value\ of\ k$ (ii) P(X< 4) (iii)P(X ≥ 4) OR Find E(X)

2	There are 500 persons of age 55 years in a town. The chance that person aged 55 years will die within next 5 years is 1%. Based on the above information, answer the following questions: (Given e-5 =0.0067)  (i) Find the probability that exactly 4 persons will die within next 5 years. (ii) Find the probability that none of the person aged 55 will die within next 5 years. (iii)Find the probability that at most 3 persons aged 55 will die within next 5 years. OR Find the probability that more than 3 persons aged 55 will die within next 5 years.

UNIT 5: INFERENTIAL STATISTICS

CONCEPT

Population: There are mainly two types of population:

(i) Finite population (ii) Infinite population

Finite population: The finite population is a group of objects, events, experiments, observation etc., which are countable. It is also known as countable population.

Infinite population: The infinite population is a group of objects events, experiments, observations etc.

Which are uncountable. It is also known as uncountable population.

Sample: A sample is a smaller group of members of a population selected to represent the population.

There are mainly two types of sampling: 1. Probability sampling 2. Non – probability sampling

Probability sampling: In this type of sampling the selection of members from the population is random such that each member of the population has an equal chance of being selected. This method is also known as random sampling.

There are mainly four types of probability sampling.

(a) Simple random sampling: In a simple random sampling every member of the population has an equal chance of being selected. This is one of the most common methods of sampling.

(b) Systematic sampling: Systematic sampling is similar to simple random sampling but is slightly easier to contact. Every member of the population is assigned a number but instead of randomly selecting numbers, the numbers are chosen at regular intervals.

(c) Stratified sampling: In stratified sampling the population is divided into subgroups called strata based on the relevant characteristic e.g. gender, age, income profession etc. The members from each subgroup are selected using simple random or systematic sampling.

(d) Cluster sampling : Cluster sampling also divides the population into subgroups but each subgroup

has similar characteristics of the whole population. This method is useful when population is dispersed.

Non- probability sampling: In this type of sampling the selection of members from the population is nonrandom and each member has not equal chance of being selected.

There are mainly four types of non- probability sampling.

(a) Convenience sampling :In convenience sampling the members are selected which are most convenient for researcher i.e. members which are easily accessible to the researcher.

(b) Voluntary response sampling :It is similar to convenience sampling .In this sampling method people who are themselves ready to conduct the survey. Collect the sample data.

(c) Judgement sampling : This type of sampling is based on the opnion of an expert. It is also called purposive sampling. However, the quality of the sample results depends on the judgement of the person selecting the sample.

(d) Snow ball sampling: In this sampling method ,we first select some persons , then with the help of selected persons we select some more and this process continues to collect the sample. It is same as a snow ball increases in size as it rolls down.

Representative and non- representative sample.

Representative sample

A representative sample is a subset of population, which accurately represents, reflects or matches the

Characteristics of the population.

Non- representative sample: A non-representative sample is a subset of a population that does not represent, reflect the characteristics of the population. It is biased in nature.

Unbiased and biased sampling

Unbiased sampling: A sampling is called unbiased sampling if each element or member of the population has an equal chance of being chosen to be the part of the sample. Simple random sampling, systematic sampling, stratified random sampling, cluster sampling are the examples of an unbiased sampling.

Biased sampling: A sampling is called biased sampling if some elements or members of the population have higher or lower probability than others, of being chosen , to be part of the sample. Convenient sampling, voluntary response sampling, judgement sampling are the examples of a biased sampling.

Parameter and statistic

Parameter: A measurable characteristics of a population is called parameter. e. g Population mean (μ),

Population variance (σ^2) , population standard deviation (σ) etc.

Statistic: A measurable characteristics of a sample is called statistic. For example sample mean ($\bar{x}$), sample variance (s^2), sample standard deviation (s) etc.

In inferential statistics, we use sample statistic to estimate the corresponding population parameter. While making an inference about the population, the parameter is known because it is impossible to collect the information from every member of the population.

Difference between population and sample:

Population	Sample
1.Population is a collection of all elements having same characteristic i.e. a population includes all the elements from a set of data.	Sample is a group of members of the population i.e. a sample consists of one or more observations Drawn from the population.
2. The measurable characteristics of population is called parameter.	The measurable characteristic of sample is called statistic.
A survey done of an entire population is accurate and more precise with no margin of error.	A survey done using a sample of the population has a margin of error.
The population focuses on the identification of the characteristic.	The sample makes inference about the population.

Central limit theorem (CLT)

The central limit theorem (CLT) states that the sampling distributing of the sample mean approaches a normal distribution (bell shaped curve) as the sample size gets larger, no matter what is the shape of the population distribution.

Statistical inference

Statistical inference is a process through which inferences about the population are made based on informations obtained from the sample.

There are mainly three types of statistical inferences:

(i)Point Estimation (ii) Interval Estimation (iii) Hypothesis testing .

Point estimation : In statistics point estimation involves the use of sample data to calculate a single value of an unknown population parameter. Single value of sample data s known as point estimate of corresponding population parameter.

For example , sample mean $\bar{x}$ is a point estimate of population meanμ, sample proportion $\bar{p}$ is a point

estimate of population proportion p.

sample mean $\bar{x} = \sum \frac{x_i}{n}$, Sample proportion (probability) $\bar{p} = \frac{x}{n}$, Sample standard deviation s $= \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$

Interval estimation

In interval estimation we find two members using sample statistics between which a population parameter lie. For example, a $< \mu < b$ is an interval estimate of the population mean μ. It indicates that population mean is greater than a but less than b.

Confidence interval : A Confidence interval is a range of values, derived from sample statistic which is likely to contain the value of an unknown population parameter.

A confidence interval consists of three parts : 1. Confidence level 2. Sample statistic 3. Margin of error

1. Confidence level : The confidence level describes the uncertainty of a sampling method i.e the probability part of a confidence interval is called confidence level. Confidence level written in decimal is called confidence coefficient.

2. Sample statistic : It is sample statistic for which the estimation of population parameter is done.

3. Margin of error: In a confidence interval , the range of values , above and below the sample statistic is called margin of error.

Interval estimation of population mean

$\mu = \bar{x} \pm$ margin of error , where $\bar{x}$ =sample mean and μ =population mean.

Calculation of margin of error (σ $known$)

The formula of calculating the margin of error is

Margin of error = $Z_{\frac{\alpha}{2}} . \frac{\sigma}{\sqrt{n}}$. Where (1- α) is called confidence coefficient and $Z_{\frac{\alpha}{2}}$ is the Z value providing an area of $\frac{\alpha}{2}$ in the upper tail of the standard normal probability distribution.

Interval estimation of population proportion

The interval estimation of population proportion is given by the following formula:

P = $\bar{p} \pm$ margin of error, where $\bar{p}$ = sample proportion and p= population proportion .

Margin of error $= Z_{\frac{\alpha}{2}} \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$.

Hypothesis testing

In hypothesis testing we begin by making a tentative assumption about population parameter called null statistics and is denoted by H_O . We then define another hypothesis called alternative hypothesis and is denoted by H_a which is of opposite of what is stated in hypothesis.

Developing null and alternative Hypothesis

1.Testing Research Hypothesis : Suppose a particular motor bike has an average fuel efficiency of 40 km/litre. Research and development of that company a new fuel injection system to increase the average fuel efficiency. H$_{O:}$ $\mu \leq 40$,H$_a$ $\mu > 40$.

2. Testing the validity of a claim : Suppose a manufacturer of soft drinks claims that two-litre soft drinks bottles has an average of atleast 150 g sugar. A sample of two-litre bottle will be selected and the contents will be measured to test the manufacture's claim. H$_O$: $\mu \geq 150$, H$_a$ $\mu < 150.\mu$

3.Testing in Decision making situations

This type of situation occurs when a decision maker must choose between two courses of action, one associated with Null hypothesis, another associated with alternative hypothesis.

H$_{O:}$ $= 3$, H$_a$:$\mu \neq 3$.This type of hypothesis test is employed in quality control procedures.

Hypothesis test for population Mean ($\sigma\ known$)

Hypothesis test may take three forms

H$_O$: $\mu \geq \mu_\circ$	H$_O$:$\mu \leq \mu_\circ$	H$_O$: $\mu = \mu_\circ$
H$_a$: $\mu < \mu_\circ$	H$_a$> $\mu_\circ$	H$_a$: $\mu \neq \mu_\circ$

The first two forms are one- tailed test and third forms is called two –tailed test.$\mu_\circ$ denote the hypothesized value of population mean μ.

t-test : A t-test is a statistical test that is used to compare the means of two groups , which may be related in certain features . It is often used in hypothesis testing. There are mainly three types of t-tests :

(a) One sample test (b) Two sample test (c) Paired t- test.

One sample test: If there is one group being compared against a standard value , perform one sample t- test e.g comparing the acidity of a liquid toa neutral pH of 7.

(ii) Two sample t- test

I f the groups come from two different population , perform two sample t-test. It is also known as independent t- test or student's t-test.

(iii) Paired t-test

If two groups come from a single population , perform paired t-test.

MULTIPLE CHOICE QUESTIONS

1.An observed set of population that has been selected for analysis is called

(a) a sample (b) a process (c) a forecast (d) a parameter

2. A Specific characteristic of a population is known as a

(a) a sample (b) parameter (c) statistic (d) mean

3. A specific characteristic of a sample is known as a

(a) population (b) parameter (c) statistic (d) mean

4. Inferential statistics is a process that involves all the following except

(a) estimating a parameter (b) estimating a statistic
(c) test a hypothesis (d) analysis relationships.

5.Which of the following statement are true?

I: The mean of the population is denoted by$\bar{x}$.
II: The population mean is a statistic.
(a) I only (b) II only (c) both I and II (d) none.

6. The following data are from a simple random sample : 5,8,10,7,10,4.

The point estimate of the population is (a) 7 (b) 6 (c) 8 (d) 9

7. A population consists of four observations 1,3,5,7. What is the variance?

(a) 2 (b) 4 (c) 5 (d) 6

8.A simple random sample consists of four observations 1,3,5,7.What is the point estimate of population standard deviation? (a) 2.3 (b) 2.58 (c) 2.87 (d) 3.1

9. A sample of 50 bulbs is taken at random. Out of 50 we found 15 bulbs are Bajaj, 17 are of Surya and 18 are of Crompton. What is the point estimate of population proportion of Surya?
(a) 0.3 (b) 0.34 (c) 0.36 (d) 0.4

10.What does it mean that you calculate a 95 %confidence interval ?

(a)The process you used will capture the true parameter 95 % of the time in long run.

(b) you can be 95% confident that your interval will include the population parameter

(c) you can be 5% confident that your interval will not include the population parameter

(d) all the above statements are true.

11.A statement made about a population parameter for testing purpose is called

(a) statistic (b) parameter (c) hypothesis (d) level of significance

12. The assumed hypothesis which is tested for rejection considering it to true is called

(a)Null hypothesis (b)alternative hypothesis (c)simple hypothesis (d)true hypothesis.

13.In testing the statistical hypothesis, which of the following statement is false ?

(a)The critical region is the values of the test statistic for which we reject the null hypothesis.

(b) The level of significance is the probability of Type – I error.

(c) In testing $H_O:\mu = \mu_\circ$,$H_a:\mu \neq \mu_\circ$ the critical region is two sided .

(d) The p – value measures the probability that the null hypothesis is true.

14.If $\propto$ = probability of type –I error , then 1-$\propto$ is

(a) Probability of rejecting H_o when H_O is true.
(b)Probability of not rejecting H_O when H_O is true.
(c) Probability of not rejecting H_O when H_a is true.
(d) Probability of rejecting H_O when H_a is true.

15.Which of the following is an assumption underlying the use of the t-distribution?

(a) The variance of the population is known
(b) The sample size are drawn from a normally distributed Population
(c)Sample standard deviation is an unbiased estimate of the population variance
(d) All the above.

16.If we reject the null hypothesis, we might be making
(a)Type –I error (b) Type –II error (c) A correct decision (d) A wrong decision

17.Which of the following is a true statement for comparing the t-distribution with standard normal distribution?

(a) The normal curve is symmetrical where as t-distributions are slightly skewed.
(b) The proportion of area beyond a specific value of 't' is ≤ the proportion of normal curve.
(c) Greater the degree of freedom , the more the t-distribution resembles the standard normal distribution.
(d)None of the above.

18. A machine makes car wheels and in a random sample of 26 wheels , the test statistic is found to be 3.07.As per t-distribution test (of 5%level of significance), what can you say about the quality of wheels Produced by the machine ? (Use t_{25} (0.05)= 2.06)

(a) Superior quality (b) Inferior quality.

(c) Same quality (d) Cannot say.

19. The population, then the degree of freedom (v) is

(a)$\frac{1}{34}$ (b) 33 (c) 34 (d) 35

20.Standard deviation of a sample from a population is called
(a) Standard error (b) Parameter(c) Statistic (d) Central limit

21.A grain wholesaler visits the granary market. While going around of rice, to make a good purchase, he takes a handful of rice from random sacks of rice, in order to inspect the quality of farmers produce. The handful rice taken from a sack of rice for quality inspection is a

(a) Statistic (b) Population (c) Parameter (d) Sample

22.In a school, a random sample of 145 students is taken to check whether a student's average calory Intake is 1500 or not. The collected data of average calories intake of sample students is presented in a frequency distribution which is called a

(a) Statistics (b) Sampling distribution (c) Parameter (d) Population sampling .

23.In a survey question for a sample of 200 individuals, 80 persons gave response 'yes', 100 gave 'NO' response and 20 gave none response.
The point estimate of the proportion who respond 'Yes'.

(a) 0.5 (b) 0.6 (c) 0.4 (d) 0.8

24. Consider the following hypothesis test: HO: p ≥ 0.75 , $H_a < 0.75$,

A sample of 300 provided a sample proportion of 0.68 .The value of the test statistic is

(a) -3.8 (b) -2.8 (c) 2.8 (d) 3.8

25. Consider the following hypothesis test : $H_O : \mu = 15, H_a : \mu \neq 15$ .A sample of 50 provided a sample mean of 14.15 . The population standard deviation is 3.The value of test statistic is

(a) – 3.2 (b) -2.003 (c) – 1.34 (d) 2.003

ANSWERS

1. (a) 2.(b) 3. (c) 4. (b) 5.(d) 6. (d) 7. (c) 8. (b) 9.(b) 10. (d)
11. (c) 12. (a) 13. (d) 14. (b) 15. (b) 16. (a) 17.(c) 18.(b) 19. (b) 20 . (c)
21. (d) 22. (b) 23.(c) 24. (b) 25. (b).

Assertion – Reason Type Questions

This type of questions consists of two statements.

Statement I is called Assertion (A) and Statement II is called Reason (R) . Read the given Statements carefully and choose the correct answer from the option given below:

(a) Both the Statements are true and Statement II is the correct explanation of Statement I.

(b) Both the Statements are true and Statement II is not the correct explanation of Statement I.

(c) Statement I is true , Statement II is false.

(d) Statement I is false , Statement II is true.

1. Statement I: A simple random sample consists of five observation 2,4,6,8,10. The point estimate of population standard deviation is $\sqrt{10}$.

Statement II : Sample standard deviation of n observations, S = $\sqrt{\sum \frac{(x_i - \bar{x})^2}{n}}$.

2. Statement I:If a sample random sample of 100 items for a population standard deviation $\sigma = 8$ resulted in sample mean x¯ =45,then95% confidence interval for population mean is (43.43, 46.57) .(Use $Z_{0.025}$ =1.96)

Statement II: Interval estimation of population mean , $\mu = \bar{x} \pm Z_{\frac{\alpha}{2}} . \frac{\sigma}{\sqrt{n}}$

3. Statement I: A refined oil manufacturing company packs the refined oil in containers labeled 16 litres and claims that mean volume of the refined oil in the containers is 16 litres with a standard deviation of 500 mL. An inspector points out doubt on its volume and tests 100 containers . As a result , he finds that mean volume is 15.9litres.So,he concludes that he was right in his suspicions.
(Given $\alpha = 0.05$,$Z_{0.05}$ =1.645)

Statement II: Rejection rule using critical value approach of upper tail test is Reject H_o if $Z \geq Z_\alpha$

4.Statement I: Rejection rule using critical value approach of two tail t- test is Reject H_O if $t \leq -t_{\frac{\alpha}{2}}$ Or if $t \geq t_{\frac{\alpha}{2}}$

Statement II: Rejection rule using critical value approach of lower tail t- test is Reject H_O if $t \leq -t_{\alpha}$.

Statement I: The critical value for $\alpha = 0.05$ with df =16 for a upper tailed test is 1.746

Statement II: The critical value for $\alpha = 0.01$ with df = 22 for a left tailed test is 4. 508

6. In a survey question for a sample of 200 individuals, 80 persons gave response ‘yes’, 100 gave response ‘no’ and 20 gave no response.

Statement I; The point estimate of the proportion in the population who respond ‘yes’ is 0.4

Statement II: The point estimate of the proportion in the population who respond ‘No’ is 0.6

7.The following data are from a simple random sample: 5,8,10,7,10,14

Statement I : The point estimate of the population mean is 9.

Statement II: The point estimate of the population standard deviation is 5.

8.Statement I: A sample of 4 students from a school was taken to see how many pens were they carrying 2, 3, 5, 6. The point estimate of the population mean is 5.

Statement II: The point estimate of the population standard deviation is 1.83.

9.A simple random sample of 400 individuals provide 100 yes responses.

Statement I : The point estimate of the population proportion that would provide Yes responses is 0.25.

Statement II: The 95%confidence interval for the population is (0.35, 0.47)

10.Consider the following hypothesis test: $H_O : \mu = 15,\ H_{a:}\mu \neq 15$. A sample of 50 provided a sample mean of 14.15. The population standard deviation is 3.

Statement I: The value of the test statistics Z = - 2.003

Statement II: The p – value is 0.008.

Answer key: 1.(c) 2. (a) 3. (a) 4. (b) 5. (c) 6 . (c) 7. (c) 8 (d) 9. (c) 10 (c)

VERY SHORT ANSWER QUESTIONS

Q.NO	QUESTIONS
1	A machine produces washers of thickness 0.50 mm. To determine whether the machine is in proper working order, a sample of 10 washers is chosen for which the mean thickness is 0.53 mm and the standard deviation is o.03 mm. Test the hypothesis at 5% level of significance that the machine is working in proper order. (Given $t_{0.025}$ = 2.262 at 9 degree of freedom). Solution: Define Null hypothesis H_0 and alternate hypothesis H_1 as follows: $H_0 : \mu = 0.50$ mm and $H_1 : \mu$ ǂ0.50 mm Thus two tailed test is applied under hypothesis H_0

	$t=\frac{\overline{X}-\mu}{s}\times\sqrt{n-1}$ = $\frac{0.53-0.50}{0.03}\times\sqrt{9}=3$ Since t(=3) > $t_{0.025}$(2.262), the null hypothesis H_0 can be rejected. Hence we conclude that machine is not working properly.
2	A group of 5 patients treated with medicine P and second group of 7 patients treated with medicine Q. The sample means were found to be 8 and 9 respectively and the sample standard deviation \`S\` is 3.71. Comment on the rejection of the hypothesis with significance level 5%. (Given $t_{10}(0.05) = 2.23$) Solution: Given: $\overline{X}=8,\ \overline{Y}=9,\ n_1=5,\ n_2=7,\ S=3.71$ $H_0 : \mu_1 = \mu_2$ and $H_1 : \mu_1 \neq \mu_2$ $t=\frac{\overline{X}-\overline{Y}}{S}\times\sqrt{\frac{n_1 n_2}{n_1+n_2}} = \frac{-1}{3.71}\times\sqrt{\frac{35}{12}} = -0.46$ Since $\lvert t \rvert < t_{10}(0.05)$, the null hypothesis H_0 is accepted at 5% level of significance.
3	The variable values in a sample of eight are recorded and their sample mean and standard deviation are found to be 0.25 and 2.659 respectively. Find the t-test value by taking the mean of the universe to be zero. Solution: Given: $\overline{X}=0.25,\ s=2.659,\ n=8,\ \mu=0$ $t=\frac{\overline{X}-\mu}{s}\times\sqrt{n} = \frac{0.25-0}{2.659}\times\sqrt{8} = \frac{0.25}{2.659}\times 2.828 = 0.2659$
4	A simple random sample of 50 items from a population with σ = 6 resulted in a sample mean of 32. Compute the 90% confidence interval for the population mean. (Given $Z_{0.05} = 1.645$) Solution: Given: $n=50,\ \sigma=6,\ \overline{x}=32$ Confidence level = 90% $1-\alpha=0.90 \Rightarrow \alpha=0.1 \Rightarrow \frac{\alpha}{2}=0.05$

	$\therefore Z_{\alpha/2} = Z_{0.05} = 1.645$ Margin of error = $Z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}} = 1.645 \times \frac{6}{\sqrt{50}}$ =1.645 × 0.848 = 1.39 μ=32 ± 1.39 Confidence interval is (32 – 1.39 , 32 +1.39)= (30.6, 33.4)
5	In a survey, the sample proportion for the population proportion is $\bar{\bar{p}}$ =0.35. How large a sample should be taken to provide a 95% confidence level with a margin of Error of 0.05 ? (Z$_{0.025}$ = 1.96) Solution: $\bar{P} = 0.35, \quad 1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$ $\therefore Z_{\alpha/2} = Z_{0.025} = 1..96$ Margin of error = $Z_{\alpha/2}\sqrt{\frac{\bar{P}(1-\bar{P})}{n}}$ 0.05 = 1.96$\sqrt{\frac{0.35\times 0.65}{n}} \Rightarrow \sqrt{n} = \frac{1.96\times 0.477}{0.05} = 18.6984$ $n = (18.6984)^2 = 349.63 \Rightarrow n = 350$
6	Ten cartons are taken at random from an automatic packing machine. The mean net weight of the ten cartons is 11.8 kg and standard deviation is 0.15 kg. Does the sample mean differ significantly from the intended mean of 12 kg ? (Given for d.f = 9, t$_{0.05}$ = 2.26) Solution: Given: $n = 10, \ \bar{X} = 11.8, \ s = 0.15, \ \mu = 12$ $t = \frac{\bar{X}-\mu}{s} \times \sqrt{n-1} = \frac{11.8-12}{0.15} \times \sqrt{9} = \frac{-0.2\times 3}{0.15} = -4$ Since $\lvert t \rvert$ > t$_{0.05}$ = 2.26, there is a significant difference between sample mean and intended mean.

7	The average height of a random sample of 400 adult males of a city is 175 cm. It is known that population standard deviation is 40. Determine the 90% confidence interval for the population mean. ($Z_{0.05}$ = 1.645) Solution: Given: $n = 400,\ \overline{\overline{x}} = 175,\ \sigma = 40$ $1 - \alpha = 0.90 \Rightarrow \alpha = 0.1 \Rightarrow \frac{\alpha}{2} = 0.05$ $\therefore Z_{\alpha/2} = Z_{0.05} = 1.645$ Margin of error = $Z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}} = 1.645 \times \frac{40}{\sqrt{400}}$ = 3.29 Confidence interval is (175 – 3.29 , 175 + 3.29)= (171.71, 178.29)
8	A sample of 100 Maruti authorized service centres showed 13 are in Delhi, 18 in Mumbai, 17 in Chennai and 15 in Kolkata. Develop an estimate of the proportion of Maruti service centres that are not in these four cities. Solution: Number of Maruti service centres that are not in these four cities = 100 – (13 + 18 + 17 + 15) = 100 – 63 = 37 The point estimate of proportion of Maruti service centres that are not in these fourcities is $\overline{P} = \frac{37}{100} = 0.37$
9	The mean and variance of a random sample of 64 observations were computed as 160 and 100 respectively. Compute the 95% confidence interval for population mean. ($Z_{0.025}$ = 1.96) Solution: Given : $n = 64,\ \overline{x} = 160,\ s^2 = 100\ \Rightarrow s = 10$ $1 - \alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$

$$\therefore Z_{\alpha/2} = Z_{0.025} = 1..96$$

Margin of error = $Z_{\alpha/2} \times \frac{s}{\sqrt{n}} = 1.96 \times \frac{10}{\sqrt{64}} =$ 2.45

Confidence interval = (160 – 2.45, 160 +2.45) = (157.55 , 162.45)

10 A fertilizer company packs the bags labelled 50 kg and claims that the mean mass of bags is 50 kg with a standard deviation of 1 kg. An inspector points out doubt on its weight and tests 25 bags. As a result, he finds that mean mass is 49.6 kg. Is the inspector right in his suspicious ? (Given $t_{24}(0.05) = 2.06$)

Solution:

$H_0 : \mu = 50$ and $H_1 : \mu \neq 50$

$$s = 1, \quad n = 25, \quad \overline{X} = 49.6$$

$$t = \frac{\overline{X} - \mu}{s} \times \sqrt{n} = \frac{49.6 - 50}{1} \times \sqrt{25} = -2$$

Since $|t| < t_{24}(0.05)$, the inspector is right in his suspicious.

SHORT ANSWER QUESTIONS

1 A 99% confidence interval for a population mean was reported to be 83 to 87. If σ=8, What sample size was used in this study ? (Given $Z_{0.005} = 2.576$)

Solution:

Given $\sigma = 8$ and confidence level = 99%

Let the sample mean be $\overline{x}$ and margin of error be E.

Then $\overline{x}$ + E = 87 and $\overline{x}$ - E = 83

By solving, we get E = 2.

Also, $1 - \alpha = 0.99 \Rightarrow \alpha = 0.01 \Rightarrow \frac{\alpha}{2} = 0.005$

$$\therefore Z_{\alpha/2} = Z_{0.005} = 2.576$$

Since E = $Z_{\alpha/2} \times \dfrac{\sigma}{\sqrt{n}} \Rightarrow\ 2 = \dfrac{2.576 \times 8}{\sqrt{n}} \Rightarrow \sqrt{n} = 10.304 \Rightarrow n = 106.172$

Therefore sample size is 106.

2 A random sample of 17 values from a normal population has a mean of 105 cm and the sum of the squares of deviations from this mean is 1225 cm^2. Is this assumption of a mean of 110 cm for the normal population reasonable? Test under 5% level of significance. (Given $t_{16}(0.05) = 2.12$)

Solution:

μ = population = 110

$\overline{X}$ = Sample mean = 105

n = sample size = 17

$$S^2 = \frac{\sum\left(X - \overline{X}\right)^2}{n} = \frac{1225}{17} = 72.0588 \ \Rightarrow S = 8.4887$$

Define Null hypothesis H_0 and alternate hypothesis H_1 as follows :

$H_0 : \mu = 110$ and $H_1 : \mu \neq 110$

$$t = \frac{\overline{X} - \mu}{s} \times \sqrt{n-1} = \frac{105 - 110}{8.4887} \times \sqrt{16} = -2.3561$$

Since $|t| > t_{16}(0.05)$, the null hypothesis H_0 can be rejected.

We may conclude that the assumption that the mean of the population is 110 cm is not valid.

3 From a sample size of 14 has 52 as mean. The sum of squares of deviations from mean is 117. Can this sample be regarded as taken from the population having 54 as mean ?

(Given $t_{13}(0.05) = 1.77$).

Solution:

μ = population = 54

$\overline{X}$ = Sample mean = 52

n = sample size = 14

$$S^2 = \frac{\sum\left(X - \overline{X}\right)^2}{n} = \frac{117}{14} = 8.357 \;\; \Rightarrow S = 2.891$$

Define Null hypothesis H_0 and alternate hypothesis H_1 as follows :

H_0 : μ = 54 and H_1 : μ ǂ 54

$$t = \frac{\overline{X} - \mu}{s} \times \sqrt{n-1} = \frac{52-54}{82.891} \times \sqrt{13} = \frac{-2 \times 3.606}{2.891} = -2.49$$

Since $|t| > t_{13}(0.05)$, the null hypothesis H_0 can be rejected.

Therefore, there is a significant difference between the sample mean and population mean.

4 Certain medicine is packed in to bags by a machine. A random sample of 10 bags drawn and their contents are found to be weigh (in kg)as follows:

40, 45, 48, 50, 52, 55, 45, 52, 63, 40

Test if the average packing can be taken to be 50 kg. (Given $t_9(0.05)$=2.26)

Solution:

H_0 : μ = 50 and H_1 : μ ǂ 50

X	40	45	48	50	52	55	45	52	63	40	Total=490
$X - \overline{X}$	–9	–4	–1	1	3	6	–4	3	14	–9	
$\left(X - \overline{X}\right)^2$	81	16	1	1	9	36	16	9	196	81	Total=446

$$\overline{X}=\frac{\sum X}{n}=\frac{490}{10}=49$$

$$S^2=\frac{\sum\left(X-\overline{X}\right)^2}{n}=\frac{446}{10}=44.6 \quad\Rightarrow S=6.678$$

$$t=\frac{\overline{X}-\mu}{s}\times\sqrt{n-1}=\frac{49-50}{6.678}\times\sqrt{9}=\frac{-1\times 3}{6.678}=-0.449=-0.45$$

Since │t│ < $t_9(0.05)$, the null hypothesis is accepted at 5% level of significance and we may conclude that the data are consistent with the average packing of 50kg.

5 Samples of sizes 10 and 14 were taken from two normal populations with S.D 3.5 and 5.2. The sample means were found to be 20.3 and 18.6. Test whether the means of the two populations are the same at 5% level . (Given $t_{22}(0.05) = 2.07$)

Solution:

$H_0 : \mu_1 = \mu_2$ and $H_1 : \mu_1 \neq \mu_2$

Given: $\overline{X}=20.3,\ \overline{Y}=18.6,\ n_1=10,\ n_2=14,\ S_1=3.5,\ S_2=5.2$

$$S=\sqrt{\frac{(n_1-1)S_1^{\ 2}+(n_2-1)S_2^{\ 2}}{n_1+n_2-2}} = \sqrt{\frac{110.25+351.52}{22}}=4.581$$

$$t=\frac{\overline{X}-\overline{Y}}{S}\times\sqrt{\frac{n_1 n_2}{n_1+n_2}} = \frac{20.3-18.6}{4.581}\times\sqrt{\frac{10\times 14}{10+14}} = \frac{1.7}{1.897}=0.896$$

Since │t│< $t_{22}(0.05)$, the null hypothesis is accepted at 5% level of significance.

6 Find the t-test value for the following two set of values:

7 , 4 , 9 , 8 and 1 , 2 , 3 , 6

Solution:

For the first data: $\overline{X}=\frac{\sum X}{n}=\frac{28}{4}=7$, For second data: $\overline{Y}=\frac{\sum Y}{n}=\frac{12}{4}=3$

Standard error:

X	$X-\overline{X}$	$\left(X-\overline{X}\right)^2$	Y	$Y-\overline{Y}$	$\left(Y-\overline{Y}\right)^2$
7	0	0	1	-2	4
4	-3	9	2	-1	1
9	2	4	3	0	0
8	1	1	6	3	9
		14			14

$$S=\sqrt{\frac{\sum\left(X-\overline{X}\right)^2+\sum\left(Y-\overline{Y}\right)^2}{n_1+n_2-2}} = \sqrt{\frac{28}{6}}=\sqrt{4.467}=2.16$$

Using t-test formula,

$$t=\frac{\overline{X}-\overline{Y}}{S}\times\sqrt{\frac{n_1 n_2}{n_1+n_2}} = \frac{7-3}{2.16}\times\sqrt{2}=\frac{4}{2.16}\times1.414 = 2.618$$

7 A simple random sample of 60 items resulted in a sample mean of 80. The population standard deviation is σ = 15.

(i) Compute the 95% confidence interval for the population mean.

(ii) Assume that sample mean was obtained from a sample of 120 items. Provide a 95% confidence interval for the population mean. ($Z_{0.025}$=1.96)

Solution:

(i) $n-60,\ \alpha=15,\ \overline{x}=80$

$$1-\alpha=0.95\Rightarrow\alpha=0.05\Rightarrow\frac{\alpha}{2}=0.025$$

$$\therefore Z_{\alpha/2}=Z_{0.025}=1..96$$

Margin of error = $Z_{\alpha/2}\times\frac{\sigma}{\sqrt{n}}=1.96\times\frac{15}{\sqrt{80}}=1.96\times1.94=3.8$

	$\mu = \overline{x} \pm 3.8$, confidence interval = (80 - .8 , 80 + 3.8) = (76.2 , 83.8) (ii) Margin of error = $Z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}} = 1.96 \times \frac{15}{\sqrt{120}} = 1.96 \times 1.37 = 2.68$ Confidence interval = (80 – 2.68 , 80 + 2.68) = (77.32, 82.68)
8	The monthly sales of a plywood shop are normally distributed with a standard deviation of Rs.900. A statistical study of sales in the last nine months has found a confidence interval for the mean of monthly sales with extremes of Rs.4663 and Rs.5839. (i) What were the average sales over the nine month period? (ii) What is the confidence level for this interval? ($Z_{0.025}$ = 1.96) Solution: (i) Let margin of error be E $\overline{x}$ + E = 5839 and $\overline{x}$ - E = 4663 By solving, we get E = 588 and $\overline{x} = 5251$ (ii) E = $Z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}} \Rightarrow 588 = Z_{\alpha/2} \times \frac{900}{\sqrt{9}} \Rightarrow Z_{\alpha/2} = \frac{588}{300} = 1.96$ $\therefore \frac{\alpha}{2} = 0.025 \Rightarrow \alpha = 0.05 \Rightarrow 1 - \alpha = 1 - 0.05 = 0.95$ Confidence level = 95%
9	According to a financial report, the majority of companies reporting profits had beaten estimates. A sample of 162 companies showed 104 beat estimates, 29 matched estimates and 29 fell short. (i) What is the point estimate of the proportion that fell short of estimates ? (ii) Determine the margin of error and provide a 95% confidence interval for the proportion that beats estimates. ($Z_{0.025}$ = 1.96) Solution : (i) Number of companies who fell short of estimates = x = 29 The point estinate of the proportion in population who fell short of estimates is $\overline{p} = \frac{x}{n} = \frac{29}{162} = 0.1790$ (ii) Number of companies that beat estimates = 104

$$\overline{p} = \frac{x}{n} = \frac{104}{162} = 0.642$$

$$1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$$

$$\therefore Z_{\alpha/2} = Z_{0.025} = 1..96$$

Margin of error = $Z_{\alpha/2}\sqrt{\dfrac{\overline{P}(1-\overline{P})}{n}}$ $=1.96\sqrt{\dfrac{0.642\times 0.358}{162}}$ = 1.96 × 0.0377 = 0.0738

Confidence interval is (0.642 – 0.0738 , 0.642 + 0.0738) = (0.5682 , 0.7158)

10 A random sample of 10 boys I.Q`s are recorded and their sample mean is found to be 97.2 and the sum of the squares of deviations from this mean is 1833.6. Do these data support the assumption of a population mean I.Q of 100 ?

(Given $t_9(0.05) = 2.26$)

Solution :

$$\overline{X} = 97.2, \quad \sum\left(X - \overline{X}\right)^2 = 1833.6$$

$$s^2 = \frac{\sum\left(X - \overline{X}\right)^2}{n} = \frac{1833.6}{10} = 183.36 \Rightarrow s = 13.54$$

Population mean μ=100

$$t = \frac{\overline{X} - \mu}{s} \times \sqrt{n-1} = \frac{97.2 - 100}{13.54} \times \sqrt{9} = \frac{-2.8}{13.54} \times 3 = -0.62$$

Since $|t| < t_{0.05}$, the null hypothesis is accepted at 5% level of significance and we may conclude that the data are consistent with the assumption of mean I.Q of 100 in the population.

Questions for practice:

Very Short answer questions:

1. The variable values ina sample of 12 are recorded and their sample mean and standard deviation are found to be 0.32 and 2.4 respectively. Find the t-test value by taking the mean of the universe to be zero. Ans.: t =0.924
2. The heights of 6 randomly chosen red roses and 12 yellow roses are recorded and their sample means are found to be 68 and 66 respectively. Also their standard deviation is found to be 3.8. Find the t-test value for the given data. Ans: t = 2.105
3. The average height of a random sample of 400 adult males of a city is 175 cm. It is known that population standard deviation is 40. Determine the 95% confidence interval for the population mean. ($Z_{0.025}$ = 1.96) Ans.: (171.08, 178.92)
4. A sample size of 10 drawn from a normal population has a mean 31 and a variance 2.25. Is it reasonable to assume that the mean of the population is 30? (Use 1% level of significance). Given that $t_9(0.01)$ =3.25. Ans.: t = 2
5. The marks of 8 randomly chosen boys and 12 randomly chosen girls are recorded and their sample means are found to be 76 and 72 respectively. Their standard deviation is found to be 3.6. Find the t-test value. Ans.: t = 2.433

Short Answer questions

1. A 95% confidence interval for a population mean was reported to be 152 to 160.
   If σ =15, what sample size was used in this study ? (Given $Z_{0.025}$ = 1.96). Ans. 54
2. A random sample of 17 values from a normal population has a mean of 105 cm and the sum of the squares of deviations from this mean is 1225 cm^2. Is this assumption of a mean of 110 cm for the normal population reasonable? Test under 1% level of significance. (Given $t_{16}(0.01)$ = 2.92)
   Ans: $|t|$=2.3561
3. From a population, a sample of 10 items having values 45, 47, 50, 52, 48, 47, 52, 49, 53, 51. Does the sample mean differ significantly from the population mean of 47.5 ? (Given $t_9(0.05)$ = 2.26) Ans: $|t|$=2.28
4. The mean of IQ`s of a group of 18 students from one area of a city was found to be 95 with a standard deviation of 10, while the mean of IQ`s of a group of 12 students from another area of the city was found to be 100 with a standard deviation of 8. Test whether, there is a significant difference between the IQ`s of two groups of students at 1% level of significant. (Given $t_{28}(0.01)$=2.763) Ans.: $|t|$=1.45

5. A survey for prices of a particular product in different stores of a city is conducted. The prices of the product in 16 different stores are given below.

   Rs.100, Rs.108, Rs.97, Rs.99, Rs.112, Rs.99, Rs.98, Rs.104, Rs.107, Rs.95, Rs.110, Rs.103, Rs.110, Rs.111, Rs.106, Rs.105.

   Assuming that prices of this product follow a normal law of variance of 25.

   (i) Compute the sample mean of the distribution.

   (ii) Determine the 95% confidence interval for the population mean. ($Z_{0.025} = 1.96$)

   Ans. (i) 104 (ii) (101.55 , 106.45)

LONG ANSWER QUESTIONS WTH SOLUTION

1. A random sample of size 16 has 53 as mean. The sum of squares of deviations from mean is 150. Can this sample be regard as taken from the population having 56 as mean? Level of significance is 5% (right tail t-test).

Ans: $\mathrm{H_o} : \mu = 56$

$\mathrm{H_1} : \mu > 56$

$n = 16,\ \bar{x} = 53,\ \mu = 56,\ \sum(x_i - \bar{x})^2 = 150$

$\therefore\ S^2 = \frac{1}{n-1} \sum_i (x_i - \bar{x})^2 = \frac{150}{15} = 10$

$\Rightarrow S = \sqrt{10} = 3.162$

$$t = \frac{\bar{x} - \mu}{(S/\sqrt{n})} = \frac{53-56}{(3.162 / \sqrt{16})} = \frac{4 \times (-3)}{3.162} = -3.795.$$

Tabulated value of $t = 1.753 \quad (df = 15,\ 0.05)$

Since the calculated value of $|t| = 3.795$ is greater than the tabulated value 1.753.

So, $\mathrm{H_o}$ can be rejected.

(Note: Area is always positive numerically on right hand side)

2. Find the t-test value for the following two samples from a population:

SET-I : 6, 4, 9, 5
SET-II : 2, 5, 6, 3

Ans: $\bar{x} = \frac{6+4+9+5}{4} = \frac{24}{4} = 6$

and $\bar{y} = \frac{2+5+6+3}{4} = \frac{16}{4} = 4$

$S^2 = \frac{1}{n_1+n_2-2}\left[\sum(x_i - \bar{x})^2 + \sum(y_i - \bar{y})^2\right]$

Now, $\sum(x_i - \bar{x})^2 = 0 + 4 + 9 + 1 = 14$

and $\sum(y_i - \bar{y})^2 = 4 + 1 + 4 + 1 = 10$

$\therefore S^2 = \frac{1}{6}[14 + 10] = 4$

$\Rightarrow S = 2$

So, we have,

$$t_{cal} = \frac{\bar{x} - \bar{y}}{S \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{6-4}{2\times\sqrt{\frac{1}{4}+\frac{1}{4}}} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2} = 1.4142 \text{ (approx.)}$$

3. A group of 5 patients treated with medicines 'P' weigh 10, 8, 12, 6, 4 kgs: Second group of 7 patients treated with medicine 'Q' weigh 14, 12, 8, 10, 6, 2,11kgs. Comment on rejection of hypothesis with significance level 5% . (**Given $\boldsymbol{t_{(10,\ 0.05)} = 1.812}$**)

Ans:

$H_o : \mu_1 = \mu_2$ and $H_1 : \mu_1 > \mu_2$

[where μ_1 *and* μ_2 denotes population means for given two groups]

Now, $\bar{x} = \frac{10+8+12+6+4}{5} = \frac{40}{5} = 8$

and $\bar{y} = \frac{14+12+8+10+6+2+11}{7} = \frac{63}{7} = 9$

Also, $S^2 = \frac{1}{n_1+n_2-2}\left[\sum(x - \bar{x})^2 + \sum(y - \bar{y})^2\right]$

Now, $\sum(x_i - \bar{x})^2 = [4 + 0 + 16 + 4 + 16] = 40$

and $\sum(y_i - \bar{y})^2 = [25 + 9 + 1 + 1 + 9 + 49 + 4] = 98$

$\therefore S^2 = \frac{1}{10}[98 + 40] = \frac{138}{10} = 13.8$

$\Rightarrow S = \sqrt{13.8} = 3.71$

$$\therefore t = \frac{\bar{x} - \bar{y}}{S \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{8-9}{3.71\times\sqrt{\frac{1}{7}+\frac{1}{5}}} = \frac{-1}{3.71 \times \sqrt{\frac{12}{35}}} = \frac{-1}{3.71\times 0.58} = -0.46$$

Given: $t_{(10,0.05)} = 1.812$

Since $t_{cal}\ value < \ t_{tab}\ value$

Hence, Null Hypothesis (H_o) may be accepted with 5% significance.

LONG ANSWER QUESTIONS FOR PRACTICE

1.Consider two in independent populations with 4 units each as follows:
Population I : 2, 4, 6, 4
Population II : 1, 2, 3, 6, 4
Find the point estimates for population means (for both I and II). Also find the standard error of the difference of two – independent sample means say $\bar{x}_1\ and\ \bar{x}_{2.}$

2. Suppose a company 'A' wants to improve sales. Past sales data indicate that the average sale was $100 per transaction. After training your sales team, recent data (taken from a sample of 25 salesmen) indicates an average sale of $ 130, with a standard deviation of $ 15. Did the training work? Test your hypothesis at a 5% alpha level (right tail test).

3. For a sample of 4 units from a population given as: 6, 8, 3, 7. If population mean is given as 6.5, comment on rejection of null hypothesis using left hand tail side with 5% level of significance.

4. The mean weekly sales of mango candy in candy stores was 146.3 mango candy per store. After an advertising campaign the mean weekly sales in 22 stores for a typical week increased to 153.7 and showed a standard deviation of 17.2. Was the advertising campaign successful? (Level of confidence 5%) [Given $t_{(21,0.05)} =$ 1.721]

5. The average number of articles produced by two machines per day are 150 and 100 with S = 18. On the basis of records of 14 day's production, can you regard both machines equally efficient at 1% level of significance?

CASE STUDY QUESTIONS WITH SOLUTION

1.A new drug is proposed to lower total cholesterol. A randomized controlled trial is designed to evaluate the effectiveness of the medication in lowering cholesterol. Thirty (30) participants are enrolled in the trial and are randomly assigned to receive either the new drug or a placebo. The participants do not know which treatment they are assigned. Each participant is asked to take the assigned treatment for six weeks. At the end of 6 weeks, each patient's total cholesterol level is measured and the sample statistics are as follows.

Treatment	**Sample Size**	**Mean**
New Drug	15	195.9
Placebo	15	227.4

with standard deviation S = 30.

Is there statistical evidence of a reduction in mean total cholesterol in patients taking the new drug for 6 weeks as compared to the participants taking placebo? (Level of significance 5%).

Ans: SET: $H_o : \mu_1 = \mu_2$ and $H_1 : \mu_1 < \mu_2$

Note: [Since $\mu_1 < \mu_2$ (for H_1) so tabulated value of t will be considered negative as its on the left hand tail while in previous questions where
$\mu_1 > \mu_2$(for H_1) is on the right hand tail so tabulated value was positive]

Given: $n_1 = n_2 = 15, \bar{x} = 195.9, \bar{y} = 227.4, \quad S = 30$

So, $t = \frac{\bar{x} - \bar{y}}{S \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{195.9 - 227.4}{30 \times \sqrt{\frac{1}{15} + \frac{1}{15}}} = \frac{-31.5}{30 \times \sqrt{\frac{2}{15}}} = -2.88$

Now, degree of freedom $= 15 + 15 - 2 = 28$

$\therefore \ t_{(28,0.05)} = -1.701$

So, we reject H_o as $-2.88 < -1.701$ [for left hand tail test]

So, we have statistically significant evidence at $\propto = 0.05$ to show that the mean cholesterol level is lower in patients taking the new drug for 6 weeks as compared to patients taking placebo.

2. A population consists of 5 units 3,7,6,8,1.

(i) Point estimate for population mean is:

a) 4 b) 5 c) 5.5 d) 4.5

(ii) Point estimate for population variance is:

a) 5.5 b) 6.5 c) 7.5 d) 8.5

(iii) Point estimate for population standard deviation is:

a) 2.91 b) 2.52 c) 2.12 d) None of these

(iv) Standard error of the sample mean is:

a) 1.20 b) 1.30 c) 1.40 d) 1.60

Ans: (i) (b) $\bar{x} = \frac{3+7+6+8+1}{5} = 5$

$\therefore$ point estimate of population mean is $(\bar{x}), i.e.,$5.

(ii) (d) $S^2 = \frac{1}{n-1} \sum_i (x_i - \bar{x})^2$
$= \frac{1}{5-1} [4 + 4 + 1 + 9 + 16]$

$= \frac{34}{4} = 8.5$

(iii) (a) Point estimate for population standard deviation is $\sqrt{8.5} = 2.91$

(iv) (b) Standard Error $= \frac{\sigma}{\sqrt{n}} = \frac{2.91}{\sqrt{5}} = \frac{2.91}{2.23} = 1.30.$

3. Consider two independent populations with 4 units each as follows:

Populations I: 4, 6, 3, 7   Populations II: 5, 6, 4, 1

(i) Point estimate for population-I variance is:

(a) $\frac{8}{3}$ (b) $\frac{10}{3}$ (c) $\frac{11}{3}$ (d) $\frac{13}{3}$

(ii) Point estimate for population-II mean is:

(a) 1   (b) 2   (c) 3   (d) 4

(iii) Point estimate for population-II variance is:

(a) $\frac{11}{3}$ (b) $\frac{13}{3}$ (c) $\frac{14}{3}$ (d) $\frac{16}{3}$

(iv) Standard error of the difference of two independent sample means (of above given two populations) is:

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\sqrt{5}$ (d) none of these

Ans: (i) (b) $\bar{x}_1$ (sample mean of population-I)

$= \frac{4+6+3+7}{4} = 5$

$\therefore\ S_1^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$

$= \frac{1}{3}(1 + 1 + 4 + 4) = \frac{10}{3}$

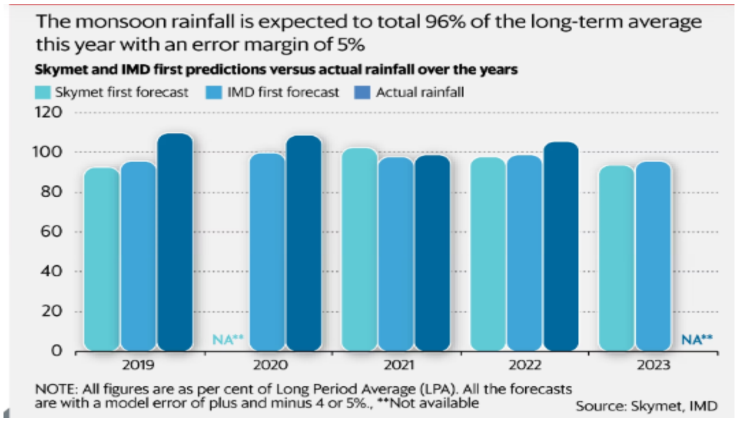
∴ Point estimate for population variance (I) is $\frac{10}{3}$

(ii) (d) $\bar{x}_2$ (sample mean of population-II)

$= \frac{5+6+4+1}{4} = 4$

∴ Point estimate for population mean is 4.

(iii)


The monsoon rainfall is expected to total 96% of the long-term average this year with an error margin of 5%

Skymet and IMD first predictions versus actual rainfall over the years

NOTE: All figures are as per cent of Long Period Average (LPA). All the forecasts are with a model error of plus and minus 4 or 5%., **Not available

Source: Skymet, IMD

(c) $S_2^2 = \frac{1}{n-1}\sum(x_i-\bar{x})^2$

$$= \frac{1}{3}(1+4+0+9) = \frac{14}{3}$$

∴ Point estimate for population variance II is $\frac{14}{3}$.

(iv) (a) S.E. $(\bar{x}_1-\bar{x}_2) = \sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} = \sqrt{\frac{\frac{10}{3}+\frac{14}{3}}{4}} = \sqrt{\frac{24}{12}} = \sqrt{2}.$

CASE STUDY QUESTIONS FOR PRACTICE

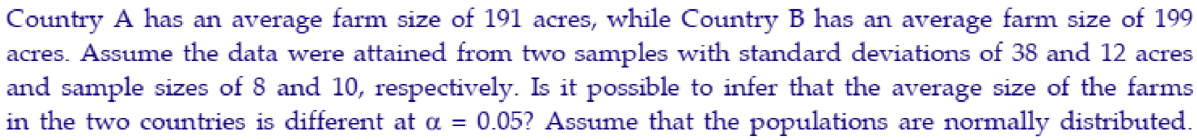
CASE STUDY 1:

Let us consider the average rainfall in a given area is 8 inches. However, a local meteorologist claims that rainfall was above average from 2016-2020 and argues that average rainfall during this period was significantly different from overall average rainfall. The following is the average rainfall for the observed period of 2016-2020:

Year	2016	2017	2018	2019	2020
Rainfall (inches)	8	5	7	5	6

1. Find the Critical value of 't' given that Standard deviation is 1.30 and $t_{0.25}$=2.776 (2)
2. Check whether the Null Hypothesis is Rejected or not give Reason? (2)

CASE STUDY 2:

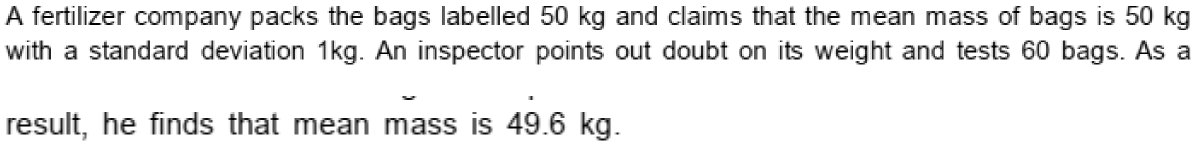


Country A has an average farm size of 191 acres, while Country B has an average farm size of 199 acres. Assume the data were attained from two samples with standard deviations of 38 and 12 acres and sample sizes of 8 and 10, respectively. Is it possible to infer that the average size of the farms in the two countries is different at $\alpha = 0.05$? Assume that the populations are normally distributed.

Based on Above Information Answer the following?

1. Find the Degree of Freedom for the sample sizes 8 and 10? (1)
2. Find the Actual 't' value for these independent Groups? Check Null Hypothesis is Rejected or not? (3)

CASE STUDY 3:

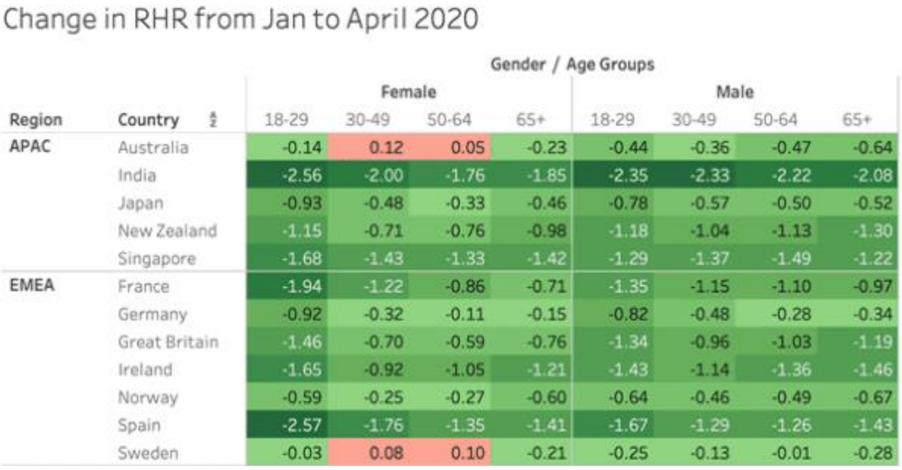


A fertilizer company packs the bags labelled 50 kg and claims that the mean mass of bags is 50 kg with a standard deviation 1kg. An inspector points out doubt on its weight and tests 60 bags. As a result, he finds that mean mass is 49.6 kg.

Based on above Information Answer the following ?

1. Find the Actual 't' value and compare with Critical 't' value? (2)
2. Check whether Null Hypothesis is Rejected or Nor ? give Reason? Take $t_{0.25}$ =2.00 (2)

CASE STUDY 4 :



Change in RHR from Jan to April 2020

		Gender / Age Groups							
		Female				Male			
Region	Country	18-29	30-49	50-64	65+	18-29	30-49	50-64	65+
APAC	Australia	-0.14	0.12	0.05	-0.23	-0.44	-0.36	-0.47	-0.64
	India	-2.56	-2.00	-1.76	-1.85	-2.35	-2.33	-2.22	-2.08
	Japan	-0.93	-0.48	-0.33	-0.46	-0.78	-0.57	-0.50	-0.52
	New Zealand	-1.15	-0.71	-0.76	-0.98	-1.18	-1.04	-1.13	-1.30
	Singapore	-1.68	-1.43	-1.33	-1.42	-1.29	-1.37	-1.49	-1.22
EMEA	France	-1.94	-1.22	-0.86	-0.71	-1.35	-1.15	-1.10	-0.97
	Germany	-0.92	-0.32	-0.11	-0.15	-0.82	-0.48	-0.28	-0.34
	Great Britain	-1.46	-0.70	-0.59	-0.76	-1.34	-0.96	-1.03	-1.19
	Ireland	-1.65	-0.92	-1.05	-1.21	-1.43	-1.14	-1.36	-1.46
	Norway	-0.59	-0.25	-0.27	-0.60	-0.64	-0.46	-0.49	-0.67
	Spain	-2.57	-1.76	-1.35	-1.41	-1.67	-1.29	-1.26	-1.43
	Sweden	-0.03	0.08	0.10	-0.21	-0.25	-0.13	-0.01	-0.28

The average heart rate for Indians is 72 beats/minute. To lower their heart rate, a group of 25 people participated in an aerobics exercise programme. The group was tested after six months to see if the group had significantly slowed their heart rate. The average heart rate for the group was 69 beats/

minute with a standard deviation of 6.5.

1. Find the Degree of Freedom of 25 people?
   (1m)
2. Calculate Actual 't' value?
   (2m)
3. Based on Null Hypothesis Test Was the aerobics program effective in lowering heart rate?
   (1m)

CASE STUDY 5:

A machine produces washers of thickness 0.50mm. To determine whether the machine is in proper working order, a sample of 10 washers is chosen for which the mean thickness is

0.53mm and the standard deviation is 0.03mm

1. Test the hypothesis at 5% level of
   significance that the machine is working in proper order.
   [Given $t_{9(0.05)} = 2.262$]
2. Find the Degree of Freedom for a sample population of 7
   members? (1)

UNIT 6: INDEX NUMBERS AND TIME BASED DATE

Meaning and Definition

Time series data::: When data of the variable is collected at distinct time intervals, for a specified period of time.

Eg: Time series analysis can be used in –

- Rainfall measurements
- Automated stock trading
- Industry forecast
- Temperature readings
- Sales forecasting

Univariate time series:::A time series in which data of only one variable is varying over time is called a univariate time series/data set

Eg: The height of ten students in a class can be recorded and this is univariate data.

Multivariate time series ::: When a time series is a collection of data for multiple variables and how they are varying over time, it is called multivariate time series/data set.

Eg: Data collected from a sensor measuring the temperature of a room every second.

Secular trend component –Secular trend component also known as trend series, is the smooth, regular and long-term variations of the series, observed over a long period of time.

Eg: Growth of population in a locality over decades .

Seasonal component – when a time series captures the periodic variability in the data, capturing the regular pattern of variability; within one-year periods.

Eg: retail sales tend to peak for the Christmas season and then decline after the holidays.

Cyclical component – when a time series shows an oscillatory movement where period of oscillation is more than a year where one complete period is called a cycle.

One example of a cyclical pattern, the business cycle, is from macroeconomics

Irregular component – these kinds of fluctuations are unaccountable, unpredictable or sometimes caused by unforeseen circumstances like – floods, natural calamities, labor strike etc.

Eg:The effects due to flood, draughts, famines, earthquakes

Trend analysis by moving average method::: This method is used to draw smooth curve for a time series data. It is mostly used for eliminating the seasonal variations for a given variable. The moving average method helps to establish a trend line by eliminating the cyclical, seasonal and random variations present in the time series. The period of the moving average depends upon the length of the time series data.

Computation of Straight-line trend by using Method of Least squares::: Method of least squares is a technique for finding the equation which best fits a given set of observations. In this technique, the sum of squares of deviations of the actual and computed values is least and eliminates personal bias.

Formulae

Fitting a straight line trend and estimating the value::

The purpose of plotting the best fitted line for trend analysis, the real values of constants 'a' and 'b' are estimated by solving the following two equations:

$\boldsymbol{\Sigma\ Y = n\,a + b\,\Sigma\ X}$

$\boldsymbol{\Sigma\ XY = a\,\Sigma\ X + b\,\Sigma\,X^2}$

Where **a** and **b**is ,$\mathbf{a}=\frac{\sum \boldsymbol{Y}}{\boldsymbol{n}}$ and $\mathbf{b}=\frac{\sum \boldsymbol{XY}}{\sum \boldsymbol{X^2}}$

'**n**' = number of years given in the data

Eg:

Given below are the consumer price index numbers (CPI) of the industrial workers.

Year	2014	2016	2017
Index Number	145	150	190

Find the best fitted trend line by the method of least squares and tabulate the trend values.

Ans:

Year	Y	X = xi – A = xi – 2017	X^2	XY	Trend value Yt = a + bX
2014	145	-3	9	-435	161.66+(-3)×(-58.5) =337.16
2016	150	-1	1	-150	161.66+(-1)×(-58.5) =220.16
2017	190	0	0	0	161.66+(0)×(-58.5) =161.66

n=3, $\sum X = -4$,$\sum XY = -585$, ,$\sum X^2 = 10$, $\sum X$=485

a=$\frac{\sum Y}{n}$ = $\frac{485}{3}$ = 161.66

b=$\frac{\sum XY}{\sum X^2}$ =$\frac{-585}{10}$ = -58.5 ,

Therefore, the required equation of the straight-line trend is given by

y = a + bx , y = 161.66-58.5x

MULTIPLE CHOICE QUESTIONS

1. **An orderly set of data arranged in accordance with their time of occurrence is called**
    - a. **Arithmetic series**
    - b. **Harmonic series**
    - c. **Geometric series**
    - d. **Time series**
2. **A time series consist of**
    - a. **Short-term variations**
    - b. **Long-term variations**
    - c. **Irregular variations**
    - d. **All the above**
3. **The secular trend is measured by the method of semi averages when:**
    - a. **Time series basedon yearly values**
    - b. **Trend is linear**
    - c. **Time series consist of even number of values**
    - d. **None of these**
4. **Wheat crops badly damaged on account of rains is**
    - a. **Cyclical movement**
    - b. **Random movement**
    - c. **Secular trend**
    - d. **seasonal movement**
5. **A time series consist seasonal variations can occur within in a period**
    - a. **Four years**
    - b. **One year**
    - c. **Two years**
    - d. **Three years**
6. **Indicate which of the following an example of seasonal variation**
    - a. **Death rate decreased due to advance in science**
    - b. **The sale of air condition increases during summer**
    - c. **Recovery in business**
    - d. **Sudden causes by wars**
7. **A fire in a factory delaying production for some weeks is**
    - a. **Secular Trend**
    - b. **Cyclical Trend**
    - c. **Irregular Trend**
    - d. **Seasonal Trend**
8. **Which of the following can’t be a component for a time series plot?**
    - a. **Seasonality**
    - b. **Trend**
    - c. **Cyclical**
    - d. **None of the above**

9. Given below are the consumer price index numbers (CPI) of the industrial workers.

Year	2014	2016	2017
Index Number	145	150	190

Find the best fitted trend line by the method of least squares

a. y = 180+25x
b. y = 180+50x
c. y = 100+25x
d. None of these

10. Seasonal variations are
a. Short term variation
b. Long term variation
c. Sudden variation
d. None

11. Increase in the number of patients in the hospital due to heat stroke is
a. Secular trend
b. Irregular variation
c. Seasonal variation
d. cyclical variation

12. Trend can be measured using by the following methods:
a. Graphical method
b. Semi averages method
c. Moving averages method
d. All of the above

13. Time reversal test of adequacy cannot be tested on
a. Lapeer’s’ method
b. Pascha’s method
c. Both a and b
d. None of these

14. When a time series is a collection of data for multiple variables and how they are varying over time, it is called
a. Univariate
b. Multivariate
c. Covariate
d. None of these

15. Examples of a time series that displays cyclical behaviour is
a. GDP
b. CPI
c. RSI
d. None of these

Solution:

1. D
   Explanation :: –A time series is a collection of data points, typically recorded at regular intervals, and organized in chronological order. This allows for the analysis of data over time.

2. D
   Explanation :: –A time series can include short-term variations, long-term variations, and irregular variations. These are different components of a time series that can help in understanding and analyzing the data.

3. B
   Explanation :: – The method of semi-averages is particularly useful when dealing with a linear trend in a time series. It helps to smooth out fluctuations and irregularities in the data, making it easier to identify the underlying linear trend.

4. D
   Explanation :: – When wheat crops are damaged due to rains, it typically represents a seasonal movement. Seasonal movements are recurring patterns that happen at regular intervals, such as due to changes in weather or seasons

5. B
   Explanation :: – Seasonal variations are patterns that repeat within a one-year period, often associated with seasonal changes like those that occur with the changing of the seasons or annual events.

6. B
   Explanation :: – when a time series captures the periodic variability in the data, capturing the regular pattern of variability; within one-year periods. The main causes of such fluctuations are usually climate changes, seasons, customs and habits which people follow at different times

7. C
   Explanation:: – These kinds of fluctuations are unaccountable, unpredictable or sometimes caused by unforeseen circumstances like – floods, natural calamities, labour strike etc.

8. D
   Explanation::-The data in any time series are based on four components are Seasonality, Trend, Cyclical and irregular compound

9. D
   Explanation::-

Year	Index number	X = xi – A = xi – 2017	X^2	XY
2014	145	-3	9	-435
2016	150	-1	1	-150
2017	190	0	0	0

n=3, $\sum X = -4$,$\sum XY = -585$, ,$\sum X^2 = 10$, $\sum X$=485

a=$\frac{\sum Y}{n}$ = $\frac{485}{3}$ = 161.66

b=$\frac{\sum XY}{\sum X^2}$ =$\frac{-585}{10}$ = -58.5 ,

Therefore, the required equation of the straight-line trend is given by

y = a + bx , y = 161.66-58.5x,

10. A

Explanation:-Long term variations can be divided into two parts: Trend or Secular Trend and Cyclical variations. Short term variations can be divided into two parts: Seasonal variations and Irregular Variations.

11. C

12. D

13. C

14. B

15. A

ASSERTION-AND-REASON TYPE QUESTIONS

Each question consists of two statements, namely,Assertion (A) and Reason

(R).For selecting the correct answer,use the following code:

(a) Both Assertion (A) and Reason (R) are the true and Reason (R) is a correct explanation of Assertion (A).

(b) Both Assertion (A) and Reason (R) are the true but Reason (R) is not a correct explanation of Assertion (A).

(c) Assertion (A) is true and Reason (R) is false.

(d) Assertion (A) is false and Reason (R) is true.

1. **Assertion (A) :**The smooth, regular and long-term variations of the series, observed over a long period of time

Reason (R) : Sunflower crops badly damaged on accounts of rain is :

2. **Assertion (A) :** The purpose of time series is to show an increasing growth pattern over time for a variable
   Reason (R): Time series analysis can be useful to see how a given asset, security, or economic variable changes over time.

3. **Assertion (A)**: In a straight line equation Y= a +bX , b is the slope .
   Reason (R) : Value of b in the trend line Y= a +bX always zero

4. **Assertion (A):** In Trend analysis by moving average method,Choice of the length of moving average is very important
   Reason (R) : Itis used to draw smooth curve for a time series data

5. **Assertion (A):** Seasonal component is a time series captures the periodic variability in the data
   Reason (R) : Death rate decreased due to advance in science

6. **Assertion (A):** The time series may reveal that the demand for ethnic clothes not only increases during Diwali but also during the wedding season.
   Reason (R) : Time series analysis helps in the measurement of financial growth.

7. **Assertion (A):** A time series can be used by management to make current decisions and plans based on long term forecasting.
   Reason (R) :A time series is a collection of data over a period of time- weekly, monthly, quarterly or yearly.

8. **Assertion (A):** The moving average of a period (extent) *m* is a series of successive averages of *m* terms at a time.
   Reason (R) : The extent of the moving average which completely eliminates the oscillatory fluctuations.

9. **Assertion (A):** The least squares method is used mostly for data fitting.
   Reason (R) : The best fit result minimizes the sum of squared errors or residuals which are said to be the differences between the observed or experimental value and corresponding fitted value given in the model.

10. **Assertion (A):**In times series data, estimating number of hotel rooms booking in next 6 months.**Reason (R) :**In times series data, Estimating the total sales in next 3 years of an insurance company

11. **Assertion (A):** Semi-average method is used for measurement of trend when trend is linear.
    Reason (R): Moving-average method is used for measurement of trend when trend in straight line.

12. **Assertion (A):**The various reasons or the forces which affect the values of an observation in a time series are the components of a time series.
    Reason (R) : The most important use of studying time series is that it helps us to predict the future behaviour of the variable based on past experience

13. **Assertion (A):** One of the most widely used in practice mathematical techniques of finding the trend values is the method of least squares.

Reason (R) : It plays an important role in finding the trend forecasts for the future economic and business time series data

14. **Assertion (A):** It is always necessary that the increase or decrease is in the same direction throughout the given period of time.
Reason (R) : The pattern of the data clustering shows the type of trend.

15. **Assertion (A):** Cyclical component is a time series shows an oscillatory movement where period of oscillation is more than a year where one complete period is called a cycle.
Reason (R) : The real Gross Domestics Product (GDP) provides good examples of a time series that displays cyclical behaviour.

Solutions:

1. (a) **Explanation:**

Assertion (A):refers to the concept of long-term variations in a time series. It suggests that when analyzing data over an extended period, we often observe patterns and variations that are regular and relatively smooth, indicating trends or changes that occur over a long time.

Reason (R) attempts to provide an example of such long-term variation. It suggests that sunflower crops being badly damaged due to rain could be considered a long-term variation in a series. This implies that over a prolonged period, there may be fluctuations in sunflower crop yields due to changing weather patterns, and this could be part of the long-term variation.

2. (a)**Explanation:**The assertion is about the purpose of time series, and the reason provides a valid explanation for the purpose of time series analysis

3. (c)**Explanation**: The value of 'b' in the linear equation represents the slope, and it's not always zero.

4. (a)**Explanation:**The assertion is true. The length of the moving average determines how much smoothing is applied to the data. A longer moving average will result in a smoother curve, but it will also be less responsive to changes in the data.

The reason is also true. Moving averages are used to smooth time series data by reducing the impact of short-term fluctuations. The longer the moving average, the smoother the curve will be.

5. (a)**Explanation::** a decrease in the death rate due to advances in science, which is not directly related to the concept of a seasonal component in a time series

6. (b)**Explanation::**A time series is a set of observations taken at specified times, usually at equal intervals. The advantages of time series analysis are reliability, seasonal pattern, estimating of trend, growth.

7. (b) **Explanation::**:Time series analysis is a specific way of analysing a sequence of data points collected over an interval of time. Concept of time series in forecasting.

8. (c)**Explanation::**Drawbacks of Moving Average is **,**the main problem is to determine the extent of the moving average which completely eliminates the oscillatory fluctuations.
9. (b)**Explanation::**Least Square is the method for finding the best fit of a set of data points. It minimizes the sum of the residuals of points from the plotted curve. It gives the trend line of best fit to a time series data. This method is most widely used in time series analysis.
10. (b)**Explanation:::** these are the example of time series problem
11. (c)
12. (b)
13. (a)
14. (d)
15. (a)

Very Short Answer (VSA) & Short Answer (SA) Questions:

1

Obtain the trend values using 3 yearly moving averages for the following data.

Year(t)	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985	1986
Y_t	66	84	90	78	96	105	93	111	87	117	114

Solution:

Year(t)	Y_t	**3yearlymovingtotal**	**3 yearly moving averages**
1976	66	-	-
1977	84	66 +84 +90=240	80
1978	90	84 +90 +78=252	84
1979	78	90 +78 +96=264	88
1980	96	78+96+106=280	93.33
1981	105	96+105+93=294	98
1082	93	105+93+111=309	103
1983	111	93+111+87=291	97
1984	87	111+87+117=315	105
1985	117	87+117+114=318	106
1986	114	-	-

2

A food processor uses a moving average to forecast next month's demand. Past actual demand (in units) is shown below. Compute a simple five-month moving average.

Month	43	44	45	46	47	48	49	50	51
Actual Demand	105	106	110	110	114	121	130	128	137

Solution:

Calculations for five-month moving average total are shown below.

MM (43-47) = $\frac{105+106+110+110+114}{5} = \frac{545}{5}$ =109.0

MM (44-48) = $\frac{106+110+110+114+121}{5} = \frac{561}{5}$ =112.2

MM (45-49) = $\frac{110+110+114+121+130}{5} = \frac{585}{5}$ =117.0

MM (46-50) = $\frac{110+114+121+130+128}{5} = \frac{603}{5}$ =120.6

MM (47-51) = $\frac{114+121+130+128+137}{5} = \frac{630}{5}$ =126.0

Month	**ActualDemand**	**5-monthMovingTotal**	**5-monthMovingAverage**
43	105	-	-
44	106	-	-
45	110	545	109.0
46	110	561	112.2
47	114	585	117.0
48	121	603	120.6
49	130	630	126.0
50	128	-	-
51	137	-	-

3 The owner of small manufacturing company has been concerned about the increase in

Manufacturing costs over the past 10 years. The following data provide a time series of the cost per unit for the company’s leading product over the past 10 years.

Year	2007	2008	2009	2010	2011
CostperUnit	332	317	357	392	402
Year	2012	2013	2014	2015	2016
CostperUnit	405	410	427	405	438

Calculatea5-yearmovingaveragefortheunitcostofthe product.

SOLUTION:

5-yearmovingtotaliscomputedasfollows:

MT(2007-2011)=332+317+357+392+402=1800

MT(2008-2012)=317+357+392+402+405=1873

MT(2009-2013)=357+392+402+405+410=1966

MT(2010-2014)=392+402+405+410+427=2036

MT(2011-2015)=402+405+410+427+405=2049

MT(2012-2016)=405+410+427+405+438=2085

Year	PerUnitCost	5-year MovingTotal	5-year MovingAverage
2007	332	-	-
2008	317	-	-
2009	357	1800	360.0
2010	392	1873	374.6
2011	402	1966	393.2
2012	405	2036	407.2
2013	410	2049	409.8
2014	427	2085	417.0
2015	405	-	-
2016	438	-	-

4

The following table shows the production of gasoline in U.S.A. for the years 1962 to 1976.

Obtain trend values for the above data using 5-yearly moving averages.

SOLUTION:

Year	**Production (in million barrels).**	**5 yearly moving total**	**5 yearly moving average**
1962	0	-	-
1963	0	-	-
1964	1	0+0+1+1+2=4	0.8
1965	1	0+1+1+2+3=7	1.4
1966	2	1+1+2+3+4=11	2.2
1967	3	1+2+3+4+5=15	3
1968	4	2+3+4+5+6=20	4
1969	5	3+4+5+6+7=25	5
1970	6	4+5+6+7+8=30	6
1971	7	5+6+7+8+9=35	7
1972	8	6+7+8+9+8=38	7.6
1973	9	7+8+9+8+9=41	8.2
1974	8	8+9+8+9+10=44	8.8
1975	9	-	-
1976	10	-	-

5

Using three yearly moving averages, determine the trend values from the following data

Year	2001	2002	2003	2004	2005	2006
Profit	142	148	154	146	157	202
Year	2007	2008	2009	2010	2011	2012
Profit	241	263	280	302	326	353

Solution:

Year	Profit	3- Yearly moving total	3 – Yearly moving trend
2001	142	-	-
2002	148	444	148
2003	154	448	149.33
2004	146	457	152.33
2005	157	505	168.33
2006	202	600	200
2007	241	706	235.33
2008	263	784	261.33
2009	280	845	281.67
2010	302	908	302.67

2011	326	981	327
2012	353	-	-

6. Fit a straight-line trend to the following data using the method of least square. Estimate the trend for 2007.

Year	2000	2001	2002	2003	2004
Sales(in tons)	10	18	33	45	63

SOLUTION:

Let T represent the year and Y represent the sales in tons year T. In the given table number of
year is n = 5 which is odd. Thus, we take middle year 2002 as origin and use transformation
u =T - 2002.
Using this transformation straight line trend is defined by the equation

$$Y = a + bu$$

$$Or \quad Y = a + b(T - 2002)$$

Usingthis transformation, weprepare following table

Year(T)	Y	u	u^2	uY
2000	10	–2	4	–20
2001	18	–1	1	–18
2002	33	0	0	0
2003	45	1	1	45
2004	63	2	4	126
$\sum$	169	0	10	133

$$Here, \sum Y = 169$$

$$\sum u^2 = 10, \sum uY = 133$$

Since, Su=0, we get $a = \frac{\sum Y}{n} = \frac{169}{5} = 33.8$

$$b = \frac{\sum uY}{\sum u^2} = \frac{133}{10} = 13.3$$

Substituting *a* and *b* in(1), we have the equation of the trend line as

Y=33.8+13.3*u*

Or Y =33.8+13.3(*T*-2002)

Trendfor2007, Y=33.8+13.3(2007-2002)

$$=33.8+13.3 \times 5$$

$$= 33.8 + 66.5$$

$$= 100.3$$

Thus,salesin2007is100.3tonnes.

7. The following table gives the production of steel (in millions of tonnes) for years 1976 to 1986.

Year	1976	1977	1978	1979	1980	1981
Production	0	4	4	2	6	8
Year	1982	1983	1984	1985	1986	
Production	5	9	4	10	10	

Fit a trend line to the above data by the method of least squares. Also, obtain the trend value for the year 1990.

SOLUTOIN:

Let T represent the year and Y represent the production of steel (in millions of tonnes) in year T. In the given table number of year is n = 11 which is odd. Thus we take middle year 1981 as origin and use transformation u = T - 1981.

Using this information, straight line trend is defined by the equation$Y=a+bu$

Or $Y=a+b(T\text{-}1981)$

Usingthistransformation,wepreparefollowingtable.

Year(*T*)	***Y***	***u***	***u2***	***uY***
1976	0	–5	25	0
1977	4	–4	16	–16
1978	4	–3	9	–12
1979	2	–2	4	–4
1980	6	–1	1	–6
1981	8	0	0	0
1982	5	1	1	5
1983	9	2	4	18
1984	4	3	9	12
1985	10	4	16	40
1986	10	5	25	50
∑	**62**	**0**	**110**	**87**

$$Here, \sum Y = 62$$

$$\sum u^2 = 110, \sum uY = 87$$

Since,$\sum u$*=0, we get* $a = \frac{\sum Y}{n} = \frac{62}{11} = 5.64$

$$b = \frac{\sum uY}{\sum u^2} = \frac{87}{110} = 0.79$$

Substituting *a* and *b* in (1)we have the equation of the trend line as

Y =5.64+0.79u

or Y =5.64+0.79(T-1981)

for the year 1990, Y = 5.64 + 0.79(1990-1981)

=5.64+0.79 X9

=12.75

Trend value for the year 1990 is 12.75 million of tonnes.

8. The following table relates to the tourist arrivals (in millions) during 2004 to 2010 in India::

Year	2004	2005	2006	2007	2008	2009	2010
Tourist arrivals	18	20	23	25	24	28	30

(I) Fit a straight-line trend by the method of least squares

(II) Estimate the number of tourists that would arrive in the year 2014.

Solution:

Let T represent the year and Y represent the tourists’ arrivals in year T. In the given table number

of year is n = 7 which is odd. Thus, we take middle year 2007 as origin and use

transformation u = T - 2007. Using this transformation, straight line trend is defined by the

equation

Y =a+bu

Or Y=a+b(T-2007)

Usingthistransformation,wepreparefollowingtable

Year(*T*)	*Y*	*u*	*u*2	*uY*
2004	18	-3	9	-54
2005	20	-2	4	-40
2006	23	-1	1	-23
2007	25	0	0	0
2008	24	1	1	24
2009	28	2	4	56
2010	30	3	9	90
∑	168	0	28	53

$$Here \sum Y = 168$$

$$\sum u^2 = 28, \sum uY = 53$$

Since, $\sum u = 0$*, we get* $a = \frac{\sum Y}{n} = \frac{168}{7} = 24$

$$b = \frac{\sum uY}{\sum u^2} = \frac{53}{28} = 1.89$$

Substituting *a* and *b* in (1)we have the equation of the trend line as

Y=24+1.89*u*

or

Y=24+1.89(*T*-2007)

Foryear2014wehave

Y =24+1.89(2014-2007)

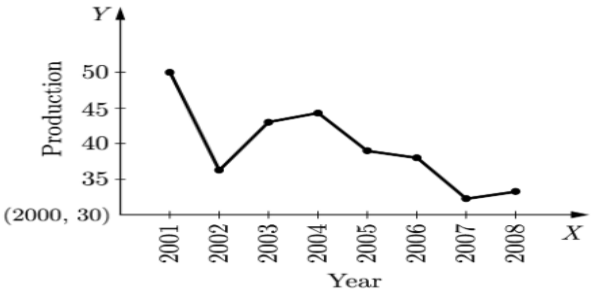
=24+1.89 X7

=37.23

9 The following table shows production (in ‘000 tonnes) of a commodity during years 2001-2008. Define the trend for the data by using graph.

Year	2001	2002	2003	2004	2005	2006	2007	2008
Production	50	36.5	43	44.5	38.9	38.1	32.6	33.7

Solution:



The above graph shows a downward trend.

10 Obtain the trend values using 4 yearly centered moving average for the following data.

Year (t)	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985
Yt	115	130	137	135	130	140	148	155	162	180

SOLUTION:

Year (t) =a	**Yt=b**	**4 yearly moving total (c)**	**4 yearly moving average (d=$\frac{c}{4}$)**	**2 unit total (e)**	**4yearly centered moving average (f=$\frac{c}{2}$)**
1976	115				
1977	130				
1978	137				
1979	135	517	129.25	262.25	131.25
1980	130	532	133	268.25	134.25
1981	140	542	135.5	237.75	136.875
1982	148	553	138.25	281.5	140.75
1983	155	573	143.25	294.5	147.25
1984	162	605	151.25	312.5	156.25
1985	180	645	161.25		

LONG ANSWER TYPE QUESTIONS

1 The average number, in lakhs, of working days lost in strikes during each year of the period 2001 – 2010 was

YEAR	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
working days lost in strikes (in lakhs)	1.5	1.8	1.9	2.2	2.6	3.7	2.2	6.4	3.6	5.4

Calculate the 3- yearly moving averages and draw the moving averages graph

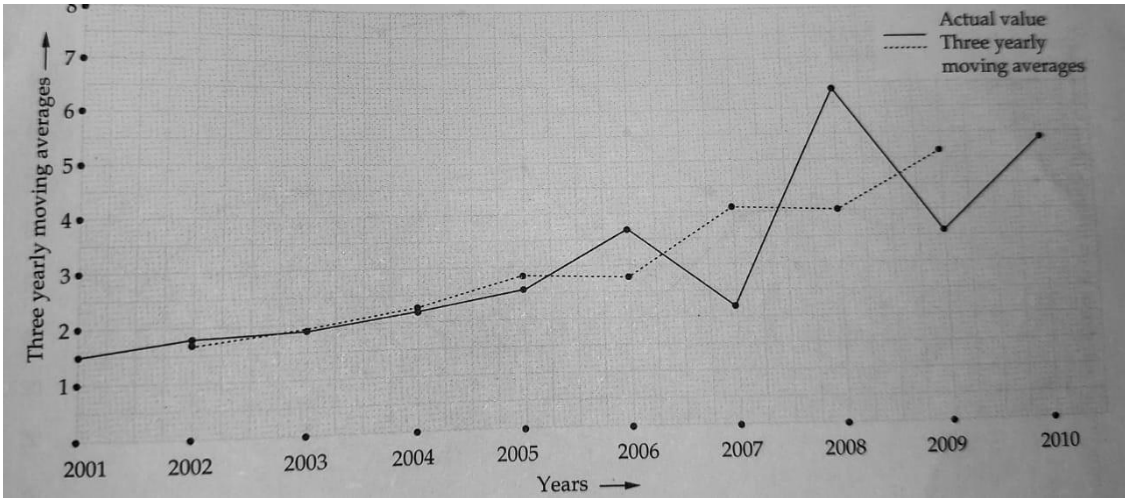
SOLUTION:

In order to calculate three moving averages, first we calculate three yearly moving totals

and place each total against the middle year of the three-year span from which the moving totals are calculated. These moving totals are given in the third column of the following table. From these three yearly moving totals, we calculate three yearly moving averages by dividing each moving total by 3 as shown in the following table

Years	**Working days lost in strikes (in lakhs)**	**Three yearly moving totals**	**Three yearly moving averages**
2001	1.5	-	-
2002	1.8	5.2	1.73
2003	1.9	5.9	1.96
2004	2.2	6.7	2.23
2005	2.6	8.5	2.83
2006	3.7	8.5	2.83
2007	2.2	12.3	4.1
2008	6.4	12.2	4.06
2009	3.6	15.4	5.13
2010	5.4	-	

The graph of these moving averages



2 Consider the following data:

Year	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
Production	137	140	34	137	151	121	124	159	157	169	172	150

Calculate a suitable moving average and show on a graph against the original data.

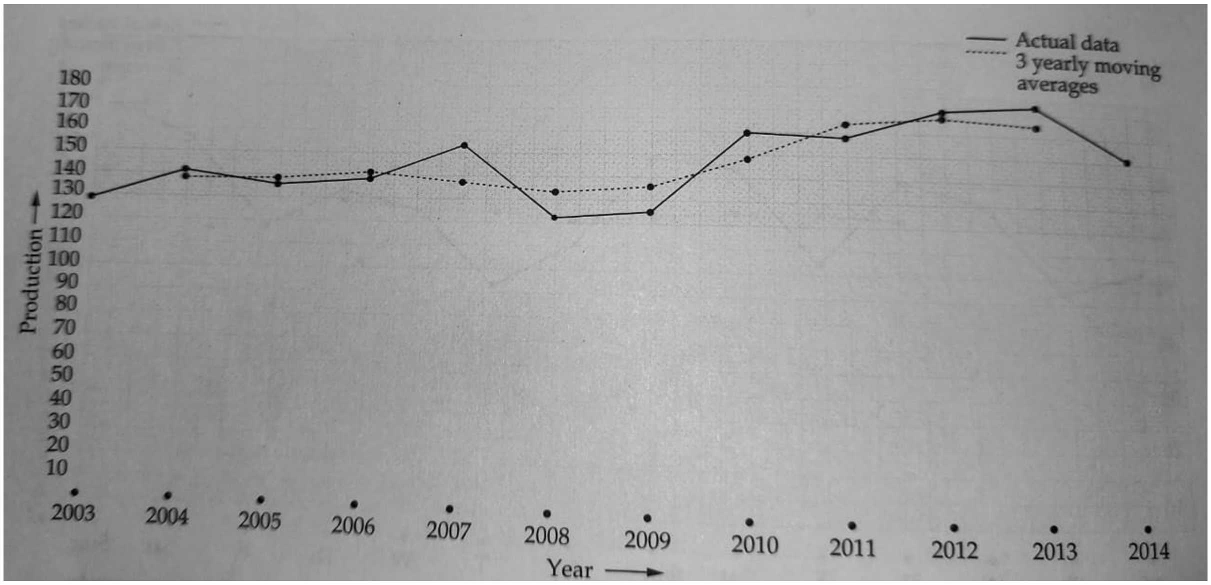
SOLUTION:

In order to find which moving average will be appropriate, we will have to estimate the length of the cycle of the above data. We observe that the data has the pattern (137, 140, 134), (137, 151, 121), (124, 159, 157), (169, 172, 150). Thus, we have cycle length of 3.

So, we will calculate 3 yearly moving averages as shown in the following table:

Year	**Production**	**3-year moving total**	**3-year moving average**
2003	137	-	-
2004	140	311	103.66
2005	34	311	103.66
2006	137	322	107.33
2007	151	409	136.33
2008	121	396	132
2009	124	404	134.66
2010	159	440	146.66

2011	157	485	161.66
2012	169	498	166
2013	172	491	163.66
2014	150	-	-



3 The table below gives the details of the electricity generated in million kilowatt hours in each quarter for the year 2020 to 2022

YEAR	QUARTER			
	1st	2nd	3d	4th
2020	8	7	6	9
2021	10	7	7	10
2022	11	7	8	10

Calculate the 4 quarterly moving average and show these moving averages on a graph

ANSWER:

The moving averages are 7.75 , 8, 8.125 , 8.375 , 8.625 , 8.75 , 8.875 , 9

4 The number of letters , in hundreds , posted in a certain city on the each day for a week is givven as follows

35, 70 ,36, 59, 62, 60, 71

Calculate the 3 days moving averages and display these graphically.

ANSWER:

The moving averages are 47 , 55.00 , 52.33, 60.33, 64.33

CASE BASED QUESTIONS

1 The annual sale of a company for 11 years is given below :

Consider the following data :

YEAR	Annual sales (in lakhs of R.S)
2007	3
2008	7
2009	2
2010	6
2011	4
2012	8
2013	3
2014	7
2015	5
2016	9
2017	4

Based on the above data answer the following :

(i) What is the 5-yearly moving averages (2marks)

(ii) What is the 6-yearly centred moving averages (2marks)

Solution:

(i)

Years	Annual sales (in lakhs of R.S)	5 yearly moving totals	5 yearly moving averages
2007	3	-	-
2008	7	-	-
2009	2	22	4.4
2010	6	27	5.4
2011	4	23	4.6
2012	8	28	5.6
2013	3	27	5.4
2014	7	32	6.4
2015	5	28	5.6
2016	9	-	-
2017	4	-	-

(ii)

Years	Annual sales (in lakhs of R.S)	6yearly moving total6	6 yearly moving averages
2007	3	-	-
2008	7	-	-
2009	2	-	-
2010	6	30	5
2011	4	30	5
2012	8	30	5
2013	3	27	4.5
2014	7	36	6
2015	5	-	-
2016	9	-	-
2017	4	-	-

UNIT – 7: FINANCIAL MATHEMATICS

Definitions and Formulas:

Perpetuity: A perpetuity is an annuity where payments continue forever.

Amount of a Perpetuity: Amount of a perpetuity is undefined since it increases beyond all bounds as time goes on.

Present value of Perpetuity: There is two types of perpetuity which are as follows:

1- The present value of a perpetuity of Rs. R payable at the end of each period, the first payment due one period hence is the sum of money which is invested now at the rate i per period will yield Rs. R at the end of each period forever. It is given by

$$R\,(1 + i)^{-1} + R\,(1 + i)^{-2} + \text{————-}$$

It is an infinite geometric series with first term R $(1+i)^{-1}$&whose common ratio is $(1+i)^{-1}$

Its sum is given by

$$=\frac{R(1+i)^{-1}}{1-(1+i)^{-1}} = \frac{R}{i}$$

Present value of a perpetuity of Rs. R payable at the end of each period, the first being due one period hence is $P = \frac{R}{i}$

where R = size of each payment

i = rate per period

2- Perpetuity of Rs. R payable at the beginning of each period, the first payment due on Present value. This annuity can be considered as an initial payment of Rs. R followed by a perpetuity of Rs. R of above type.

Thus, the present value is given by $R + \frac{R}{i}$

R = size of each payment

i = rate per period

Sinking Fund:

A sinking fund is a fund that is accumulated for the purpose of paying off a financial obligations (future capital expense, or repayment of a long-term debt)at some future designated date.

The periodic payments of Rs. R made at the end of each period required to accumulate a sum of Rs. A over n periods with interest charged at the rate i per period is

$$R = \frac{A}{S_{\overline{n}|i}}$$

Where $S_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$

R = Size of each instalment or payment
i = rate per period
n = number of instalments
A = lumpsum amount to be accumulated

Difference between Sinking Fund and Savings Account

Sinking fund and savings account, both, involve setting aside an amount of money for the future.The main difference is that the sinking fund is set up for a particular purpose and is to be used at a particular time, while the savings account is set up for any purpose that it may serve.

VALUATION OF BONDS

Bond: It is a written contract between a borrower and a lender (bond holder). Through this contract, the borrower promises to pay a specified sum at a specific future date and to pay interest payments at a specific rate at equal intervals of time until the bond is redeemed (repaid).

A bond is characterized by following terms:

Face Value: The face value (also known as par value) of a bond is the price at which the bond is sold to buyers (investors) at the time of issue. It is also the price at which the bond is redeemed at maturity. It is also known as the par value of the bond.

Redemption Price: It is the amount the bond issuer pays at maturity. It is usually equal to the face value in case the bond is redeemed at **par**.

Discount: Where the market price of bond is less than its face value (par value), the bond is selling at a **discount.**

Premium: if the market price of bond is greater than its face value, the bond is selling at a **premium.**

Bond valuation: is the determination of the fair price of a bond. As with any security or capital investment, the theoretical fair value of a bond is the present value of the stream of cash flows itis expected to generate. Hence, the value of a bond is obtained by discounting the bond’s expected cash flows to the present using an appropriate discount rate.

Nominal rate of interest: It is the rate at which a bond yields interest. It’s also known as coupon rate.

Coupon Rate: A bond’s coupon rate denotes the annual interest rate paid by the bond issuer to the bond holder. It is simply the coupon payment C as a percentage of the face value F. Coupon yield is also called nominal yield.

$$\text{Coupon rate} = \frac{C}{F}$$

Current Yield: The current yield is simply the coupon payment C as a percentage of the (current)bond price P_o.

$$\text{Current yield} = \frac{C}{P_0}$$

Yield to Maturity (YTM): The yield to maturity (YTM) is the discount rate which returns themarket price of a bond without embedded optionally, it is identical to required return. YTM is thusthe internal rate of return of an investment in the bond made at the observed price. Since YTM canbe used to price a bond, bond prices are often quoted in terms of YTM.

To achieve a return equal to YTM, the bond owner must:

- Buy the bond at a price Po
- Hold the bond until maturity
- Redeem the bond at par

Relationship between Bond, YTM and Coupon yield

- When a bond sells at a discount, YTM > current yield > coupon yield.
- When a bond sells at a premium, coupon yield > current yield > YTM
- When a bond sells at par, YTM = current yield = coupon yield

Present Value Approach (or bonds with a maturity period)

In the present value approach, we first calculate the present value of each expected cash flowand then we add all the individual present values to obtain the value or fair value ofpurchase price of a bond.Let there be a bond where

Value of bond or Market price of bond or, purchase price of bond = V

Face Value = F

Redemption price or Maturity value = C

Number of cash flows or number of periodic payments = n

i_d be the rate of interest per period

Let periodic dividend payment R (periodic – interest) is given by,

$$R = C \times i_d$$

Let Yield rate or interest rate per period = i

Coupon payment or periodic interest (dividend) payment = R

Present value of annuity of periodic dividend payments of R for n periods is given by

$P1=R \left[\frac{1-(1+i)^{-n}}{i}\right]$

The present value of redemption price of the bond is given by:

$P2=C\ (1+i)^{-n}$

Let V be the purchase price of the bond, then

V = P1 + P2

$V = R\left[\frac{1-(1+i)^{-n}}{i}\right] + C\,(1+i)^{-n}$

Calculation of EMI or Amortization of Loans

Principal: It is the initial amount of money borrowed (or invested).

Interest: It is the price paid by a borrower for the use of lender's money. It is the differencebetween initial amount borrowed and end payment made to the lender.

Rate of Interest: It is the percentage of the sum borrowed which is charged for a defined lengthof time for using the principal generally on a yearly basis.

Term of Loan: It is the defined length of time it will take for a loan to be completely paid offwhen the borrower is making regular payments.

Meaning of EMI

EMI stands for equated monthly instalment. It is a monthly payment that we make towards alone we opted for at a fixed date of every month.

A loan is said to be amortized if it can be discharged by a sequence of equal payments (EMI)made over equal periods of time. Each payment can be considered as consisting of two parts:

(i) Interest on the outstanding loan, and

(ii) Repayment of part of the loan

Thus, a loan is amortized when part of each periodic payment is used to pay interest and theremaining part is used to reduce the principal.

Methods of calculation of EMI or Instalment

EMI or Instalment can be calculated by two methods:

1. Flat Rate Method
2. Reducing-balance method or Amortization of Loan

Flat Rate Method: - In the flat-rate method, each interest charge is calculated based on theoriginal loan amount, even though the loan balance outstanding is gradually being paid down. TheEMI amount is calculated by adding the total principal of the loan and the total interest on theprincipal together, then dividing the sum by the number of EMI payments, which is the number ofmonths during the loan term. Let P, I and n be the principal of the loan, the total interest on the principal and number ofmonths in loan period respectively. EMI is given by the formula

$EMI = \left(\frac{P + I}{n}\right)$

Reducing-Balance Method or Amortization Formulas :

When one is amortizing a loan, at the beginning of any period, the principal outstanding is the present value of the remaining payments. Using this fact, we obtain the formulas in table that describe the amortization of an interest-bearing loan of Rs. P, at a rate i per period by n equal payments of Rs. R each and such that a payment is made at the end of each period.

Table-Amortization Formulas

1. Periodic payment or Instalment

$$R = P\left(\frac{i}{1-(1+i)^{-n}}\right) \qquad \frac{P}{a_{n\neg i}}$$

2. Principal outstanding at the beginning of kth period =

$$= R\left[\frac{1-(1+i)^{-n+k-1}}{i}\right]$$

3. Total interest paid = nR - P

Where P= amount of the loan

R= size of equal payment

i = rate per period

n = number of equal payments

Nominal and Effective Rate of Interest:

Nominal Rate of Interest:

The announced or stated rate of interest is called nominal rate of interest.

Effective Rate of Interest:

The actual rate by which the money grows during each year is called the effective rate of interest.

Relation between effective rate of interest and nominal rate of interest:

$$r_{eff} = \left(1 + \frac{r}{m}\right)^{m} - 1$$

where r_{eff} = effective rate of interest

r = nominal rate of interest

m = number of conversion periods per year

In case of continuous compounding of nominal rate r, the effective rate of interest is

$r_{eff} = e^{r} - 1$

Compound Annual Growth Rate

Meaning of Compound Annual Growth Rate

Compound annual growth rate (CAGR) depicts the cumulative performance of a particular variable over a period of time via compounding effect. It is often used to evaluate the performance of different investments by an individual or enterprise through annual rate of return. The basic concept of compound growth rate can be explained with the help of following example:

If you had invested Rs.1,000, and it grew at a compound rate of 10% annually,

Year 1: Rs. 1,000 + (1,000 x 10%) = Rs.1,100

Year 2: Rs.1,100 + (1,100 x 10%) = Rs.1,210

Year 3: Rs.1,210 + (1,210 x 10%) = Rs.1,331

Year 4: Rs.1,331 + (1,331 * 10%) = Rs.1,464.10

So, the amount would be worth Rs.1,464 after 4 years.

Formula for calculation of CAGR

CAGR = $[\left(\frac{EV}{SV}\right)^{\frac{1}{n}} - \mathbf{1}]$ **x 100**

where: EV = Investment's ending value

SV = Investment's starting value

n = Number of investment periods (months, years, etc)

IMPORTANT POINTS

- CAGR is expressed in percentage
- CAGR can be used to compare historical returns in different investment portfolios
- CAGR eliminates the effects of volatility on periodic investments

Difference between Average Annual Growth rate and Compound Annual Growth Rate

Average Annual Growth Rate is calculated by dividing the cumulative return by the number of years. It usually inflates the results. Compound Annual Growth Rate is determined by compounding effect on the return or any variable taken into consideration. Many investors prefer CAGR because it smoothens out the volatile nature of year-by-year growth rates and provides more accurate measure of performance as compared to Average Annual Growth rate.

Use of Compound Annual Growth Rate

The CAGR can be used to calculate the average growth of a single investment. As we know, due to market volatility, the year-to-year growth of an investment is likely to appear uneven. For example, an investment may increase in value by 9% in one year, decrease in value by 3% the second year and increase in value by 5% in the next. CAGR helps smooth returns when growth rates are expected to be volatile and inconsistent.

CAGR is also used to track the performance of various business measures of one or multiple companies alongside one another. For example, over a five-year period, a Retail Store's market share CAGR was 1.75%, but its customer satisfaction CAGR for the same period was -0.51%. Thus, comparing the CAGRs of measures within a company reveals its strengths and weaknesses.

Depreciation

The decrease in the value of the assets such as building machinery and equipment of all kinds is called depreciation.

Scrap value, Residual value or salvage value:

The value of a depreciable asset at the end of its useful life is called the scrap value.

Total depreciation or wearing value:

The difference between the original cost and the scrap value is called total depreciation.

Book value:

The difference between the original cost of the asset and the accumulated depreciation at any given date is called the book value of that asset on that date

Methods of computing the annual depreciation:

There are three methods of computing the annual depreciation:

1. Straight line method
2. Sum of the years digit method
3. Written down value method or reducing balance method

Linear or Straight line method:

The linear method of depreciation is the simplest and the most widely used method to calculate the depreciation for fixed assets. Buildings, machinery, computer, automobiles, electronic items are examples of assets that will last for more than one year, but will not last indefinitely. Value of such assets decreases year by year because of passage of time, wear and tear, outdated, accidents etc. The work efficiency of asset decreases and expenses on repairs increases.

According to this method the annual depreciation is given by

$$D = \frac{C - S}{n}$$

Where D = the annual depreciation

C = the original cost of the asset

S = estimated scrap value or salvage value

n = the useful life in years

Remark: In the above formula, C-S is the total depreciation.

It should be noted that:

1. When rate of depreciation is given with the words per annum (e.g. 10% p.a.) and the date of acquisition is given then Depreciation is charged only for the period for which the asset is held.

2. When the date of acquisition is not given, then depreciation is charged for full year.

3. When rate of depreciation is given without the words per annum, then depreciation is charged for the full year.

(i) It is a simple method of calculating the Depreciation.

(ii) In this method, asset can be depreciated up to the estimated scrap value.

(iii) In this method, it is easy to know the amount of Depreciation as it is uniform every year.

Sum of the years digit method:

In this method, the fraction of the asset to be depreciated each year is obtained by putting the digit of the year in reverse order over the sum of the digits of the life periods. A greater fraction of the cost of the asset is depreciated in the earlier years of the life of the asset.

Written down value method or reducing balance method:

This method is called the constant percentage method or diminishing balance method. In this method, the annual depreciation is a constant percentage of the book value of the depreciated asset at the end of the preceding year.

This constant percentage must be determined so that the book value of the asset at the end of its estimated life is reduced to scrap value. The book value at the end of the nth year is given by

$S = C (1-r)^n$

Where, S = Book value at the end of nth year

C = original cost of the asset

r = rate of depreciation

MULTIPLE CHOICE QUESTIONS

1	The present value of a sequence of payments of Rs 60 made at the end of each 6 months and continuing forever, if money is worth 4% p.a. compounded semi-annually is a) Rs 3,000 b) Rs 3,500 c) Rs 4,000 d) Rs 4,500 **Answer: a) Rs 3,000** R = 60, $i = \frac{0.04}{2} = 0.02$

	Present value of Perpetuity $P = \frac{R}{i} = \frac{60}{0.02} = 3000$
2	At 6% converted quarterly, find the present value of a perpetuity of Rs 600 payable at thebeginning of each quarter. a) Rs 30,400 b) Rs 35,500 c) Rs 40,600 d) Rs 45,000 **Answer: a) Rs 40,600** R = 600, i = $\frac{0.06}{4}$ = 0.015 $\because P = R + \frac{R}{i} = 600 + \frac{600}{0.015} = 600 + 40{,}000 = 40{,}600$
3	EMI stands for a) Equated Monthly Installments b) Emerging Monthly Installments c) Easy Monthly Installments d) None of the above **Answer: a) Equated Monthly Installments**
4	At what rate converted semi-annually will the present value of a perpetuity of Rs 450 payable at the end of each 6 months be Rs 20,000 ? a) 4.0% b) 4.5% c) 5.0% d) 5.5% **Answer: b) 4.5%** **P** = 20,000, R = 450 $\because i = \frac{R}{P} = \frac{450}{20{,}000} = 0.0225 \Rightarrow \frac{r}{2} = 2.25\% \Rightarrow r = 4.5\%$
5	The effective rate which is equivalent to a stated rate of 6% compounded semi-annually is a) 0.0609 b) 0.9061 c) 0.0062 d) 0.9601 **Answer: a) 0.0609** r = 6%, i = $\frac{0.06}{2}$ = 0.03, n = 2 half years

	$\because r_{eff} = (1+i)^n - 1 = (1+0.03)^2 - 1 = 0.0609$
6	A machine costing Rs 50,000 has a useful life of 4 years. The estimated scarp value is Rs 10,000, then the annual depreciation is a) Rs 20,000 b) Rs 10,000 c) Rs 5,000 d) Rs 2,500 **Answer: b) Rs 10,000** Annual Depreciation (D) = $\frac{\text{Original value} - \text{Scrap value}}{\text{useful life}} = \frac{50{,}000 - 10{,}000}{4} = 10{,}000$
7	CAGR stands for a) Compound Aggregate Growth Rate b) Compound Annual Growth Rate c) Computed Annual Growth Rate d) Computed Aggregate Growth Rate **Answer: b) Compound Annual Growth Rate**
8	If rate of the return on an investment is positive, then it indicates a) Profit b) Loss c) No Profit No Loss d) None **Answer: a) Profit**
9	Which of the following is correct a) Present value of Bond = Present value of Interest payments – Present value of Maturity Payments b) Present value of Bond = Present value of Interest payments + Present value of Maturity Payments c) Present value of Bond = Present value of Interest payments x Present value of Maturity Payments d) None of these **Answer: (b)**
10	A bond is said to be issued at premium when

	a) Coupon rate > Required returns b) Coupon rate = Required returns c) Coupon rate d) Current rate **Answer: a) Coupon rate > Required returns**
11	Which of the following is a series of constant cash flows that occurs at the end of each period for some fixed number of periods? a) Ordinary annuity b) Annuity due c) Perpetuity d) option a and c **Answer: a) Ordinary annuity**
12	Mr. X takes a loan of Rs 2,00,000 with 10% annual interest rate for 5 years. Calculate EMI under Flat Rate system. a) Rs 4,000 b) Rs 5,000 c) Rs 6,000 d) Rs 7,000 **Answer: b) Rs 5,000** $I = \frac{PRT}{100} = \frac{2,00,000 \text{ x } 10 \text{ x } 5}{100} = 1,00,000$ $EMI = \frac{P + I}{n} = \frac{2,00,000 + 1,00,00}{5 \text{ x } 12} = 5000$
13	The deprecation remains constant according to which method? a) Sum of years of digits b) Units of production c) Declining Balance d) Straight line method **Answer: d) Straight line method**
14	Amortization is related to a) Intangible assets b) Tangible fixed assets c) Any fixed assets d) none of these **Answer: a) Intangible assets**
15	Depreciation is a) an increase in the value of an asset overtime.

	b) Resource diminished over the long run because of utilization. c) Asset that can quickly be turned into cash. d) Possession of assets over liabilities. **Answer: b) Resource diminished over the long run because of utilization.**

ASSERTION REASONING BASED QUESTIONS

In the following questions, a statement of assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not the correct explanation of A.

(c) A is true but R is false.

(d) A is false but R is true

1	**Assertion (A):** If the nominal rate of interest is 12.5% and inflation is 2%, then the effective rate of interest is 10.5%. **Reason (R):**If the nominal rate is calculated only at the end of a year, then the effective rate of interest is same as the nominal rate of interest. **Answer: (b)** r_{eff} = Nominal rate – inflation = 12.5% - 2% = 10.5% $r_{eff} = (1 + \frac{r}{m})^{m} – 1$, if m = 1 then $r_{eff} = r$
2	**Assertion (A):**The rate of interest for which the present value of perpetuity of Rs 500 payable at the end of each quarter be Rs 40,000 is 5%. **Reason (R):** Applying formula $P = \frac{R}{i}$ **Answer: (a)** **P** = 40,000, R = 500 $\because i = \frac{R}{P} = \frac{500}{40,000} = 0.0125 \Rightarrow \frac{r}{4} = 1.25\% \Rightarrow r = 5\%$

3	**Assertion (A):**Mr Roy purchased a house for Rs 2,00,000 in 2012. After some years he sells his house for Rs 2,50,000, then the rate of return is 25%. **Reason (R):**Rate of return = $\frac{\text{Current Value}-\text{Original Value}}{\text{Original Value}}$ x 100 % **Answer: (a)** Rate of return = $\frac{2,50,000-2,00,00}{2,00,000}$ x 100 % = 25%
4	**Assertion (A):** If the rate of return on an investment is negative, then it indicates profit. **Reason (R):** If the rate of return on an investment is negative, then it indicates loss. **Answer: (d)**
5	**Assertion (A):**An annuity in which the periodic payment begins on a fixed date and continue forever is called perpetuity. **Reason (R):** The amount or future value of perpetuity is defined. **Answer: (c)**
6	**Assertion (A)**: An investment of Rs 10,000 becomes Rs 50,000 in 4 years, then the CAGR (Compounded Annual Growth Rate) is given by [$^{4}\sqrt{5}$ – 1] x 100. **Reason (R):** CAGR = [$(\frac{\text{Ending Value}}{\text{Starting Value}})^{\frac{1}{n}}$ – 1] x 100 **Answer: (a)** CAGR = [$(\frac{\text{Ending Value}}{\text{Starting Value}})^{\frac{1}{n}}$ – 1] x 100 = [$(\frac{50,000}{10,000})^{\frac{1}{4}}$ – 1] x 100 = [$^{4}\sqrt{5}$ – 1] x 100.
7	**Assertion (A):**The effective rate, equivalent to a nominal rate of 8% p.a. compounded semi-annually is 7.24% **Reason (R):** If r is the nominal rate and m is the number of conversion periods per year, then effective rate of interest = $(1 + \frac{r}{m})^{m} - 1$ **Answer: (d)** effective rate of interest = $(1 + \frac{0.08}{2})^{2} - 1 = (1.04)^{2} - 1 = 0.0816$ = 8.16%
8	**Assertion (A):** A machine costing Rs 50,000 has a useful life of 4 years. The estimate scarp value is Rs 10,000, then the annual depreciation is Rs 10,000.

	Reason (R): Annual depreciation = $\frac{\text{Original Value}-\text{Scarp Value}}{\text{useful life}}$ **Answer: (a)** Annual depreciation = $\frac{50{,}000-10{,}000}{4}$ = 10,000
9	**Assertion (A):** The present value of perpetuity of Rs 3120 payable at the beginning of each year, if money is worth 6% effective is Rs 55,000. **Reason (R):** The present value of perpetuity of Rs 3120 payable at the beginning of each year, if money is worth 6% effective is Rs 55,120. **Answer: (d)** R = 3120, i = 6% = 0.06 $\because P = R + \frac{R}{i} = 3120 + \frac{3120}{0.06} = 3120 + 52{,}000 = 55{,}120$
10	**Assertion (A):** A bond is said to be issued at Discount whenCoupon rate < Required returns **Reason (R):** A bond is said to be issued at Discount whenCoupon rate > Required returns **Answer: (c)**

VERY SHORT ANSWER TYPE QUESTIONS

1	Find the present value of a sequence of payments of 60 made at the end of each 6 months and continuing forever, if money is worth 4% compounded semi-annually. **Solution:** This is a perpetuity of type (i), since payments are made at the end of each period. given that R =60 and i = 0.04/2 = 0.02 T then present value of a perpetuity P = $\frac{R}{i} = \frac{60}{0.02} = 3000\ R.S$
2	At 6% converted quarterly, find the present value of a perpetuity of 600 payable at the end of each quarter.

	Solution: Given that R = 600, i = 0.06/4= 0.015 Then the present value of a perpetuity P = $\frac{R}{i} = \frac{600}{0.015} = 40000\ R.S$
3	At what rate converted semi-annually will the present value of a perpetuity of 450 payable at the end of each 6 months be R.S 20000 ? **Solution:** let r be the interest rate converted semi-annually. Then i, the interest rate per period is $\frac{r}{2}$ Since P = $\frac{R}{i}$ where P = 20,000 and R = 450 we have i = $\frac{R}{p} = \frac{450}{20000}$ = 0.0225 $\frac{r}{2}$= 0.0225 R = 0.045 or 4.5 %
4	How much money is needed to endure a series of lectures costing 2500 at the beginning of each year indefinitely, if money is worth 3% compounded annually? **Solution:** We have R= 2500, i = 0.03 Money needed to endure a series of lectures costing R.S 2500 at the beginning of each year means the present value of a perpetuity of R.S 2500 payable at the beginning of each year P = R + $\frac{R}{i}$ = 2500 + $\frac{2500}{0.03}$ = R.S 85833.33
5	The present value of a perpetual income of x at the end of each six months is 40000. Find the value of x if money is worth 6% compounded semi-annually. **Solution:** We have P = 40,000

	$i = \frac{0.06}{2} = 0.03$ We know that $P = \frac{R}{i} = 40{,}000$ $40000 = \frac{X}{0.03}$ **x = R.S 1200**
6	A R.S 2,000, 8% bond is redeemable at the end of 10 years at R.S 105. Find the purchase price to yield 10% effective rate . **ANSWER** : the present value of the bond is R.S 1,792
7	Mr X plans to save amount for higher studies of his son, required after 10 years. He expects the cost of these studies to be R.S 1,00,000. How much should he save at the beginning of each year to accumulate this amount at the end of 10 years, if the interest rate is 12% compounded annually ? **ANSWER:** = R.S 5087.87

SHORT ANSWER TYPE QUESTIONS

1	A company establishes sinking fund to provide for the payment of R.S 1,00,000 debt. maturing in 4 years. Contributions to the fund are to be made at the end of every year. Find the amount of each annual deposit if interest is 18% per annum. **Solution:** let each annual deposit to the sinking fund be R.S R. Then R is given by $A = RS_{n\neg i} = \frac{R[(1+i)^n - 1)}{i}$ $100000 = \frac{R[(1+0.18)^4 - 1)}{0.18}$ $= \frac{R(0.9388)}{0.18} = R\,(5.2156)$ $R = \frac{1000000}{5.2156} = R.S\ 19173.25$

2	7 In 10 years, a machine costing R.S 40,000 will have a salvage value of R.S 4,000. A New Machine at that time is expected to sell for R.S 52,000. In order to provide funds for the difference between the replacement cost and the salvage cost, a sinking fund is set up into which equal payments are placed at the end of each year. If the fund earns interest at the rate 7% compounded annually, how much should each payment be? **Solution:** Amount needed after 10 years = Replacement Cost – Salvage cost = 52000 – 4000 = 48000 The payments into sinking fund consisting of 10 annual payments at the rate 7% per year is given by $$\boldsymbol{A = RS_{n\neg i} = \frac{R[(1+i)^{n} - 1)}{i}}$$ **480000 =** $\boldsymbol{\frac{R[(1+0.07)^{10}-1)}{0.07}}$ **R =** $\boldsymbol{\frac{48000}{13.8164480} = R.S\ 3474.12}$
3	Mr. X takes a loan of R.S 2,00,000 with 10% annual interest rate for 5 years. Calculate EMI under Flat Rate system. **Solution:** We are given that P = R.S 2,00,000 I = $\frac{10}{100}(200000)(5) = 100000$ n = 5 years = 5x12 = 60 EMI is given by the formula $$EMI = \frac{P+I}{n}$$ $EMI = \frac{200000+100000}{60}$ = R.S 5000

4 A couple wishes to purchase a house for R.S 10,00,000 with a down payment of R.S 2,00,000. If they can amortize the balance at 9% per annum compounded monthly for 25 years, what is their monthly payment? What is the total interest ?

Given $\boldsymbol{a_{300\neg0.0075=119.1616}}$

Solution: The monthly payment R needed to pay off the balance 8,00,000 at 9% per annum compounded monthly for 25 years (300 months) is given by

$$\mathrm{R} = \frac{\mathrm{P}}{(a_{\neg i})}$$

$$= \frac{\mathbf{800000}}{\boldsymbol{a_{300\neg0.0075}}} = \frac{\mathbf{800000}}{\mathbf{119.1616}} = R.S\ 6713.57$$

The total interest paid = nR – P

= (6713.57)(300) - 800000

= R.S 1214071

5 Find the effective rate of interest equivalent to a nominal rate of 6% compounded (i) Semi-annually (ii) Quarterly (iii) Continuously

Solution:

(i) When compounded semi-annually

We have r = 0.06 and m = 2

$$r_{eff} = \left(1 + \frac{r}{m}\right)^{m} - 1 = \left(1 + \frac{0.06}{2}\right)^{2} - 1 = 0.0609 = 6.09\%$$

(ii) When compounded quarterly

We have r = 0.06 and m = 4

$$r_{eff} = \left(1 + \frac{r}{m}\right)^{m} - 1 = \left(1 + \frac{0.06}{4}\right)^{2} - 1 = 0.0613 = 6.13\%$$

(iii) When compounded continuously

$$r_{eff} = e^{r} - 1 = e^{0.06} - 1 = 0.0618 = 6.18\%$$

6.	What effective rate is equivalent to a nominal rate of 8% converted quarterly? **ANSWER :** the effective rate is 8.24%. This means that the rate 8.24% compounded annually yields the same interest as the nominal rate 8% compounded quarterly.
7	Consider a bond with a coupon rate of 10% charged annually. The par value is R.S 2,000 and the bond has 5 years to maturity. The yield to maturity is 11 %. What is the value of the bond ? **ANSWER :** the value of the bond is 1,927

LONG ANSWER TYPE QUESTIONS

1	A company sets aside a sum of Rs 10000/- at the end of each year in a sinking fund so that at the end of 20 years it would amount to a balance sufficient to repay the machinery. Assuming that the cost of machinery remains constant at the end of 20 years and that money earns 10% p.a. compound interest, find the cost of the machinery. If the number of years is 10 instead of 20 then what is the cost of the machinery? **Solution**: $A=\frac{R[(1+i)^{n}-1]}{i}$ R=10000, n=20, I =10% p.a. $A=\frac{10000[(1+0.1)^{20}-1]}{0.1}$ Let $x=1.1^{20}$ logx=20×log1.1=20×0.0414=0.828 x=antilog(0.828)=6.730 A=10000×57.3 Cost of the machinery=Rs 573000/- If n = 10 then $A=\frac{10000[(1+0.1)^{10}-1]}{0.1}$ **(Hint:1.1^{10} = 2.594)** 10000×15.94=159400 Therefore the new cost of machinery=Rs 1,59,400/-

2 Mrs A takes a loan of Rs.5,00,000/- from a bank at the rate of 6% p.a. for 2 years. Calculate her EMI using flat rate method.Also she purchased a car worth Rs,5,00,000/- and paid Rs.1,00,000/- as cash down payment and balance in equal monthly installments in 2 years. If bank charges 6% p.a. compounded monthly , calculate the EMI.

Solution:

EMI= $\frac{P+Pni}{n}$,Here P = 5,00,000 , n= 12×2 = 24 months , i=$\frac{6}{12\times 00}$ = 0.005

EMI =$\frac{5,00,000+5,00,000\times 24\times 0.005}{24}$ =$\frac{5,60,000}{24}$ = 23,333.33/-

Also EMI = $\frac{Pi(1+i)^n}{(1+i)^n-1}$ Here P=5,00,000 – 1,00,000 = 4,00,000

i=0.005 , n= 12×2 = 24 months

EMI =$\frac{4,00,000\times 0.005(1+0.005)^{24}}{(1+0.005)^{24}-1}$

HINT: $(1.005)^{24}$ = 1.123

So EMI = $\frac{2000\times 1123}{123}$ = 18,260

NEW EMI = Rs.18,260/-

3 Suppose Mr.X invested Rs.1,00,000/- in a mutual fund and the value of the investment at the time of redemption was Rs.1,50,000/-. If CAGR for the investment is 8% , calculate the number of years for which he has invested the amount. If CAGRis 4 % what is the number of years of investment ?

Solution:

$$\text{CAGR\%}=\left[\left(\frac{FV}{PV}\right)^{1/n}-1\right]\times 100$$

$$8=\left[\left(\frac{1,50,000}{1,00,000}\right)^{1/n}-1\right]\times 100$$

$$\frac{8}{100}=\left[\left(\frac{3}{2}\right)^{1/n}-1\right]$$

$$\left(\frac{3}{2}\right)^{1/n}=1.08$$

	Taking log $\frac{1}{n}[log3 - log2] = log1.08$ n =$\frac{log3-log2}{log1.08}$=$\frac{1761}{334}$ logn =log1761-log334 logn =0.7221 n= antilog(0.7221)=5.273 n(approx.)=5 years ii)If CAGR is 4% , then 0.04 =$\left(\frac{3}{2}\right)^{1/n}$-1 By taking log and antilog in $\left(\frac{3}{2}\right)^{1/n}$=1.04 , we get n=1.036 n(approx.)=1 year
4	A firm sets aside a sum of Rs 50,000/- at the end of each year , in a sinking fund so that at the end of 15 years it would amount to a balance , sufficient to replace the machinery. Assuming that the cost of machinery remains constant at the end of 15 years and that money earns 12% p.a compound interest . If the number of years is 20 instead of 15 , what is the difference between the cost of machinery in both the cases.[**Hint:** $\mathbf{1.12^{15} = 5.47, 1, 12^{20} = 9.638}$**]** **ANS: i) Rs18 ,62,500/- ii) Rs17,36,650/-**
5	Mr X takes a loan of Rs 50,00,000/- from a bank at the rate of 12% p.a. for 15 years . Calculate his EMI using flat rate method . After that he took a vehicle loan worth Rs 50 lakh and paid Rs 5,00,000/- as cash down payment and balance in equal monthly installment in 15 years if bank charges 12% p.a. compounded monthly . Calculate the EMI.**[Hint:**$\mathbf{1.01^{30} = 1.346}$] **ANS: Rs 2,16,667/- and Rs 1,74,825/-**

6 Suppose Mrs. Y invested Rs 6,00,000/- in a mutual fund and the value of investment at the time of redemption was Rs 8,00,000/-. If CAGR for the investment is 10% , calculate the number of years for which he has invested the amount .If the CAGR is 20% then find the difference between the number of years of investment for both the situations.

ANS: n= 3years(app.)

And , difference = 1 year

(CASE BASED QESTIONS 2+2=4 marks)

1 I am doing a sequence of payments of Rs. 40,000 at the end of every 6 months and continuing forever, if money is worth 16% p.a compounded semi-annually.

(i) Find the present value of the sequence of payments.

(ii) At what rate of interest will the present value of a perpetuity of Rs.5,00,000/- payable at the end of every 6 months be Rs 80 lakhs ?

Solution: (i) P=$\frac{R}{i}$. Here i=$\frac{16/2}{100}$=0.08

P==$\frac{40000}{0.08}$=5,00,000

So Present value= Rs. 5,00,000/-

(ii)i=$\frac{r}{100} = \frac{500000}{8000000}=\frac{5}{80}$

r=$\frac{50}{8}$%

2 I plan to save amount for higher studies of my daughter, required after 5 years I expect the cost of these studies to be Rs. 6 lakhs.

(i) How much should I save at the beginning of each year to accumulate this amount at the end of 5 years, if the interest rate is 10% compounded annually?

(ii) If the interest rate is 6% p.a compounded annually, then what amount should I save at the beginning of each year.

Solution:

i) $A = \frac{R(1+i)[(1+i)^n - 1]}{i}$

$A = \frac{R[(1+i)^{n+1} - (1+i)]}{i}$ …….(1)

$6{,}00{,}000 = \frac{R[(1+0.1)^6 - (1+0.1)]}{0.1}$

$R = \frac{60000}{1.1^6 - 1.1} = 89418.78$ **[Hint:(1.1^6=1.772)]**

Therefore I has to save at the beginning Rs89419(approx.)

ii) From eqn(1) we get $R = \frac{Ai}{[(1+i)^{n+1} - (1+i)]}$

$R = \frac{600000 \times 0.06}{[(1.06)^6 - (1.06)]}$ **[Hint :$\mathbf{1.06^6 = 1.419}$]**

R=1,00,200

Therefore , I have to save at the beginning Rs 100200/-(approx.)

3 Examination In-charge of our school buys a copier machine for Rs. 2 lakhs. She estimates that she can use this machine for 6 years and that the machine will only be worth Rs. 10,000/- at the end of its life.

(i) Determine annual depreciation cost for her responsibility under linear method of depreciation.

(ii) Prepare a depreciation schedule for the copier machine.

Solution:

i) $D = \frac{C-S}{n}$, where C= original cost , S= scrap value , n=useful life

Here C=2,00,000/- , S=10,000/- and n=6 years

D=31666.67

Therefore annual depreciating cost for her responsibility is Rs31667(approx.)

YEAR	BOOKVALUE (BEGINNING OF EACH YEAR)	DEPRECIATION	BOOKVALUE (AT THE END OF EACH YEAR)
1	2,00,000	31667	168333
2	168333	31667	136666
3	136666	31667	104999
4	104999	31667	73332
5	73332	31667	41665
6	41665	31667	9998

4 i) What sum of money 1 need to invest now so as to get Rs10,000/- at the beginning of every month forever, if the money is worth 15% p.a. compounded monthly

ii)Also find the present value of my sequence of payments of Rs20,000/- made at the end of every six months and continuing forever, if money is worth 15% p.a. compounded quarterly.

ANS) i) Rs 8,10,000/- ii)Rs 5,33,300/-

5 i) I have an initial investment of Rs2,00,000/- in an investment plan. After 2 years it will be grown to Rs3,00,000/- . find the expected rate of return.

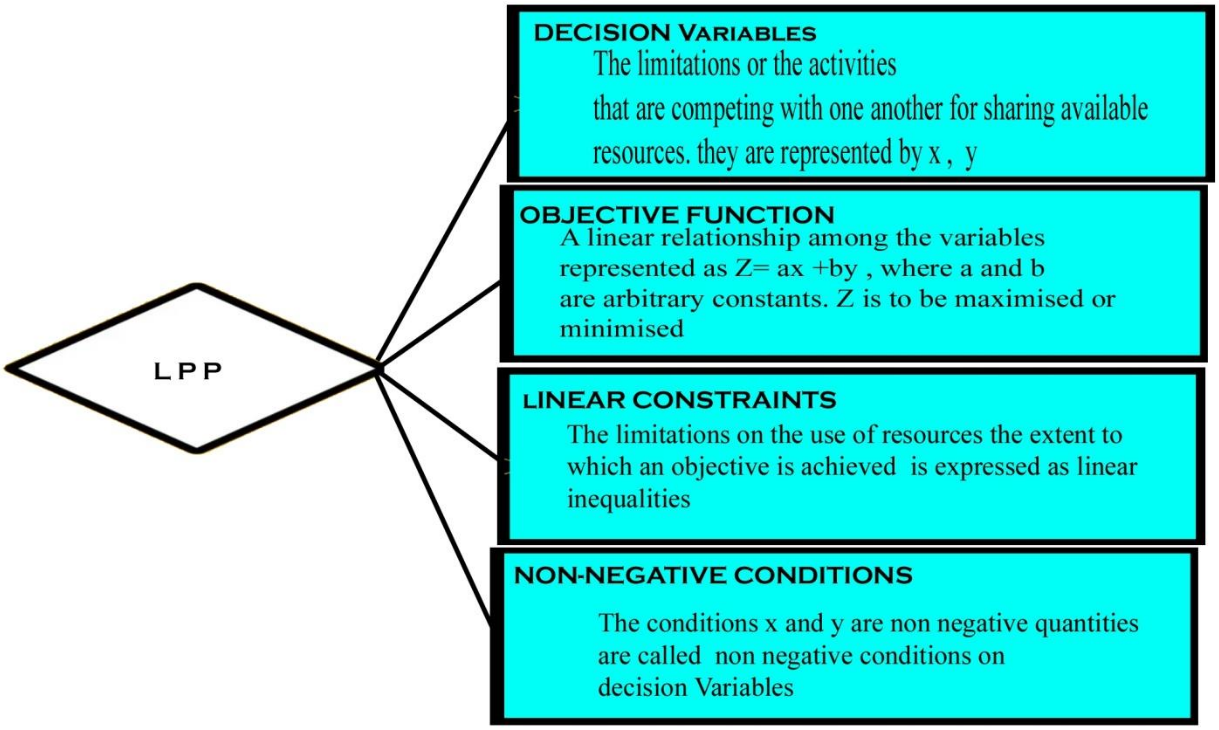
ii) but if I invest Rs2,00,000/- in 10 % of Rs100 shares of a company available at a premium of Rs 25, Find my nominal rate of return

ANS) i) 50% ii) 8%

UNIT 8: LINEAR PROGRAMMING

Linear Programming is one of the techniques for determining an ***optimal solution*** of interdependent constraint and factors in view of the available resources.

A Linear programming problem is one that is concerned with finding optimal value of a linear function of several variables subject to a number of conditions on the variables in the form of Linear inequations or equations in the variables involved



DECISION VARIABLES
The limitations or the activities that are competing with one another for sharing available resources. they are represented by x , y

OBJECTIVE FUNCTION
A linear relationship among the variables represented as Z= ax +by , where a and b are arbitrary constants. Z is to be maximised or minimised

LINEAR CONSTRAINTS
The limitations on the use of resources the extent to which an objective is achieved is expressed as linear inequalities

NON-NEGATIVE CONDITIONS
The conditions x and y are non negative quantities are called non negative conditions on decision Variables

Here the word Linear indicates that all inequations or equations used in the problem are linear.

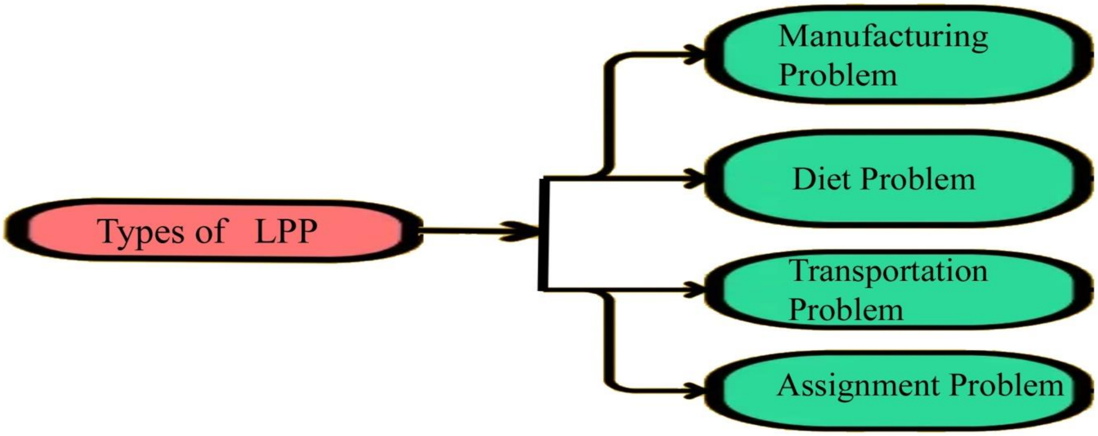
The process of converting an optimization problem in Mathematical terms is called ***Mathematical Formulation*** of LPP.

The set of values of decision variables which satisfy the constraints of LPP is called the ***Solution of LPP.***

The common region determined by all the constraints including the non negative constraints of a linear programming problem is called the feasible region or solution region for the problem .

Points within and on the boundary of the feasible region represent feasible solution of the constraints . Any point outside the region is an infeasible solution

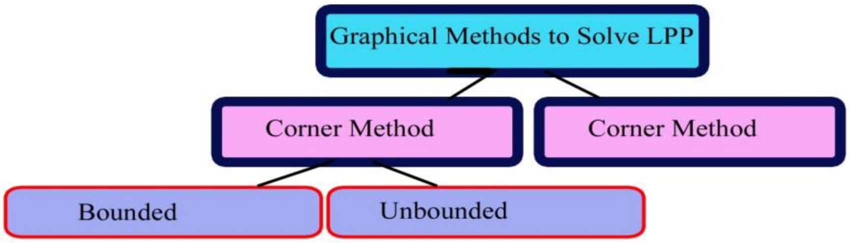
Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called the optimal solution .



The following Theorems are fundamental in solving linear programming problems:

Theorem 1: Let R be the feasible region (convex polygon) for a linear programming problem and let Z = ax + by be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2: Let R be the feasible region for a linear programming problem, and let Z = ax + by be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) ofR. If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, it must occur at a corner point of R.



Corner point method for solving a linear programming problem:

The method comprises of the following steps:

(i) Find the feasible region of the linear programming problem and determine its corner points (vertices).

(ii) Evaluate the objective function Z = ax + by at each corner point. Let M and m respectively be the largest and smallest values at these points.

(iii) If the feasible region is **bounded**, M and m respectively are the maximum and minimum values of the objective function.

If the feasible region is **unbounded**, then

(i) M is the maximum value of the objective function, if the open half plane determined by ax + by > M has no point in common with the feasible region. Otherwise, the objective function has no maximum value.

(ii) m is the minimum value of the objective function, if the open half plane determined by ax + by < m has no point in common with the feasible region. Otherwise, the objective function has no minimum value.

If two corner points of the feasible region are both optimal solutions of the same type, i.e., both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type

To Solve using graphical method we use

1. Corner point Method

Or

2. Iso – Profit or Iso –cost method.

Steps to solve LPP by Corner Point Method:

1. Find feasible region on the graph and determine its corner points (vertices).
2. Evaluate the objective function Z=ax+ by at each corner point. Let M and m, respectively denote the largest and smallest values of these points.
3. (i) When the feasible region is bounded, M and m are the maximum and minimum values of Z.

(ii) In case, the feasible region is unbounded, we have:

(a) M is the maximum value of Z, if the open half plane determined by ax+ by >M has no point in common with the feasible region. Otherwise, Z has no maximum value.

(b) Similarly, m is the minimum value of Z, if the open half plane determined by ax+ by<m has no point in common with the feasible region. Otherwise, Z has no minimum value.

Steps to solve ISO-Profit/ ISO-Cost Method:

1. As discussed in Corner-Point method, identify the feasible region and extreme (corner) points.
2. Give some convenient values to Z and draw the line so obtained in xy- plane.
3. If the objective function is to be maximized, then draw lines parallel to the line in step 2.
4. Obtain a line which is farthest from the origin and has at least one point common to the feasible region.
5. If the objective function is to be minimized, then draw lines parallel to the line in step 2 and obtain a line which is nearest to the origin and has at least one point common to the feasible region.
6. Find the co-ordinates of the common point obtained in step 4(in case of maximum) or step 5(in case of minimum). The point so obtained determines the optimal solution and the value of the objective function at these points give the optimal solution.

MULTIPLE CHOICE QUESTIONS

1	Corner points of the feasible region for an LPP are (0, 2), (3, 0), (6, 0), (6, 8) and (0, 5). Let F = 4x+ 6y be the objective function. The Minimum value of F occurs at .... (a) only (0, 2) (b) only (3, 0) (c) the mid-point of the line segment joining the points (0, 2) and (3, 0) only (d) any point on the line segment joining the points (0, 2) and (3, 0). **ANSWER : d** **Explanation:** We can evaluate F at each of the corner points to see which one gives us the minimum value. • At (0, 2), F = 4(0) + 6(2) = 12. • At (3, 0), F = 4(3) + 6(0) = 12. • At (6, 0), F = 4(6) + 6(0) = 24. • At (6, 8), F = 4(6) + 6(8) = 72. • At (0, 5), F = 4(0) + 6(5) = 30. We can see that the minimum value of F is 12, which occurs at the points (0, 2) and (3, 0). Therefore, the minimum value of F occurs at any point on the line segment joining the points (0, 2) and (3, 0). In other words, the set of all optimal solutions to this LPP is a line segment.
2	Solution set of the inequality 2x+ y> 5 is ....

(a) The half plane containing origin

(b) The open half plane not containing origin

(c) xy- plane excepts the points on the line 2x+ y= 5

(d) None of these

ANSWER : b

Explanation:

(0,0) does not satisfy $2x+ y> 5$.

3 The point at which the maximum value of $Z = 3x+ 2y$ subject to the constraints

$x+ 2y\leq 2$, $x \geq 0$, $y \geq 0$ is ....

(a) (0, 0)   (b) (1.5, – 1.5)   (c) (2, 0)   (d) (0, 2)

ANSWER : c

Explanation:

The corner points of the feasible region are (0, 0), (2, 0), and (0, 1). We can evaluate Z at each of these points to find the maximum value.

- At (0, 0), $Z=3(0)+2(0)=0$.
- At (2, 0), $Z=3(2)+2(0)=6$.
- At (0, 1), $Z=3(0)+2(1)=2$.

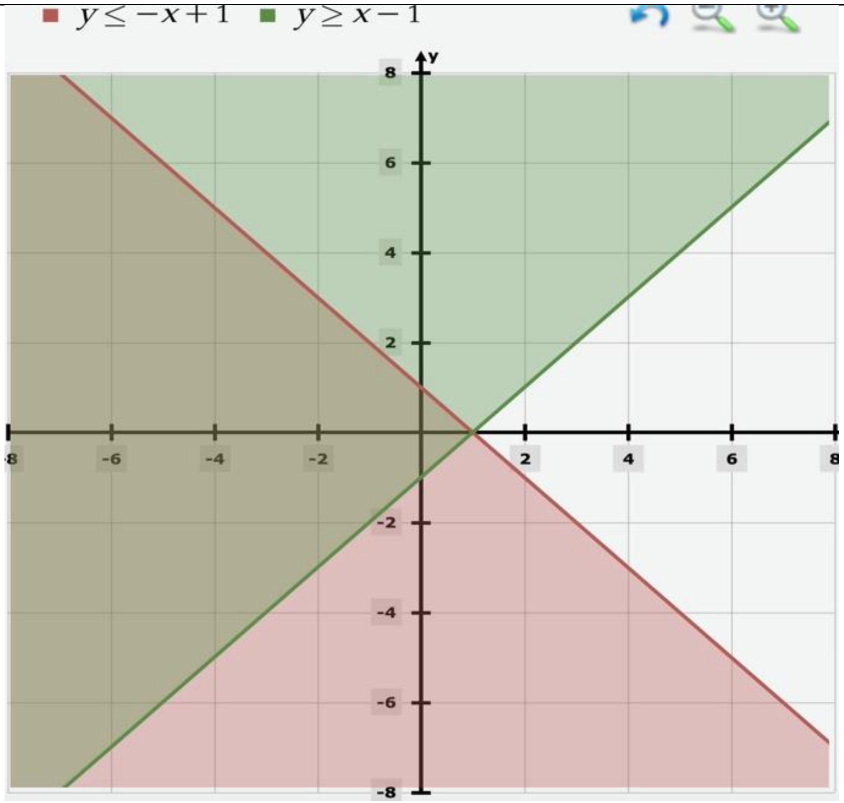
Therefore, the maximum value of Z is 6, which occurs at the point (2, 0).

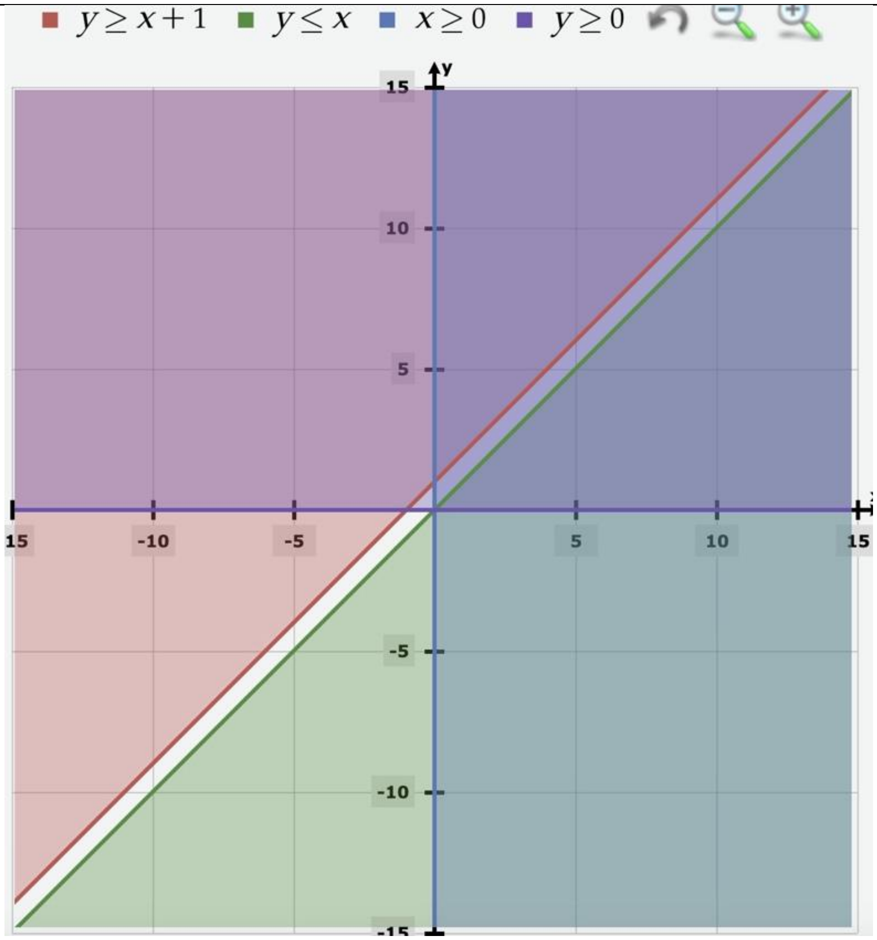
4 The feasible region of the inequality $x +y \leq 1$ and $x – y \leq 1$ lies in quadrants.

(a) Only I and II   (b) Only I and III   (c) Only II and III   (d) All the four

ANSWER : d

Explanation:

	
5	The region represented by the in equation $x - y \leq -1$, $x - y \geq 0$, $x \geq 0$, $y \geq 0$ is .... (a) bounded (b) unbounded (c) do not exist (d) triangular region **<u>ANSWER : c</u>** **Explanation:**

	
6	The conditions $x \geq 0, y \geq 0$ are called: a) Restrictions only b) non-negative restrictions c) Negative restrictions d) None of these **ANSWER: b** **Explanation:** The conditions $X \geq 0 \; and \; Y \geq 0$ are called non-negative restrictions.
7	In Linear Programming Problems, the restrictions or limitations under which the Objective function is to be optimized are called (a) Constraints (b) Objective functions (c) Decision variables (d) None of the above **ANSWER : b**

Explanation:

The objective function is the function that is to be optimized, either maximized or minimized. The decision variables are the variables that are being optimized.

EXERCISE (UNSOLVED)

1 Objective function of an LP problem is :

(a) a constant (b) a function to be optimized

(c) an inequality (d) None

ANSWER : b

2 The optimal value of the objective function is attained at the points

(a) Given by the intersection of the lines representing inequations with axes only.

(b) Given by the intersection of the lines representing inequations with X- Axis only.

(c) Given by the feasible points of the feasible region

(d) At the Origin.

ANSWER : c

3 For the LP problem maximize Z = 2x + 3y. The co-ordinates of the corner point

Of the bounded feasible region are A (3, 3), B (20, 3), C (20, 10), D (18, 12),E (12, 12).

The Maximum value of Z is

(a) 72 (b) 80 (c)82 (d)70

ANSWER : a

ASSERTION REASONING QUESTIONS

For the following questions, two statements are given – one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes (i), (ii), (iii) and (iv) as given below:

(a) Both A and R are true and R is the correct explanation of the assertion

(b)Both A and R are true but R is not the correct explanation of the assertion

(c) A is true, but R is false

(d) A is false, but R is true

1 **Assertion (A) :** Feasible region is the set of points which satisfy all of the given constraints

Reason (R): The optimal value of the objective function is attained at the points on X-axis

Only

ANSWER : c

Explanation:

Assertion (A) is true. The feasible region is indeed the set of points that satisfy all of the given constraints. It's where all the constraints are simultaneously met.

Reason (R) is false. The optimal value of the objective function is not necessarily attained at the points on the X-axis only.

2 **Assertion (A):** It is necessary to find objective function value at every point in the

Feasibleregion to find optimum value of the objective function.

Reason(R): For the constrains$2x+3y \leq 6$, $5x+3y \leq 15$, $x \geq 0$ and $y \geq 0$ corner

points of the feasible region are (0, 2), (0, 0) and (3, 0).

ANSWER :d

Explanation:

- Assertion (A) is false. It is not necessary to find the objective function value at every point in the feasible region to find the optimum value of the objective function. The optimum value can be found by evaluating the objective function at the corner points of the feasible region.

- Reason (R) is true. The corner points of the feasible region are indeed (0, 2), (0, 0), and (3, 0), which are the points where the constraints $2x + 3y \leq 6$, $5x + 3y \leq 15, x \geq 0, and\ y \geq 0$ intersect.

3 **Assertion (A):** The maximum value of Z = 11x+7y

Subject to the constraints are

$2x+y \leq 6, x \leq 2, x, y \geq 0$.

Occurs at the point (0,6).

Reason (R): If the feasible region of the given LPP is bounded, then the maximum

and minimum values of the objective function occurs at corner points.

ANSWER :a

Explanation:

- Assertion (A) is true. The maximum value of the objective function Z, subject to the given constraints, indeed occurs at the point (0, 6) when Z = 11x + 7y. You can verify this by evaluating the objective function at this point.

- Reason (R) is also true. If the feasible region of a linear programming problem is bounded, the maximum and minimum values of the objective function typically occur at the corner points of the feasible region. This is a fundamental principle in linear programming. So, the Reason correctly explains why the maximum value of Z occurs at a corner point (0, 6) in this case.

4 **Assertion (A):** Consider the linear programming problem.

Maximize Z = 4x+y

Subject to constraints $x+\ y^{\leq}50;\ x+\ y\ ^{\geq}100\ and\ x,y\ ^{\geq}\ 0$. Then,

maximum value of Z is 50

Reason (R): If the shaded region is bounded then maximum value of objective

function can be determined.

<u>**ANSWER :d**</u>

Explanation:

- Assertion (A) is false. Because feasible region does not exist

Reason (R) is true. If the shaded region in the feasible region is bounded, then it's typically possible to determine the maximum value of the objective function.

5 **Assertion (A):**The linear programming problem, maximize Z = x+2y

subject to constraints $x-y\leq\ 10{,}2x\ +\ 3y\leq\ 20\ and\ x\geq\ 0;\ y\geq 0$. It gives the maximum value of Z as $\frac{40}{3}=13.33$

Reason (R): To obtain maximum value of Z, we need to compare value of Z at

all the corner points of the shaded region.

<u>**ANSWER :a**</u>

Explanation:

The corner points of the feasible region are (0, 0), (10, 0), (0, 6.67), and (3.33, 3.33). We can evaluate the objective function at each of these points to find the maximum value of Z.

Corner point	x	y	Z = x + 2y
(0, 0)	0	0	0
(10, 0)	10	0	10
(0, 6.67)	0	6.67	13.33
(3.33, 3.33)	3.33	3.33	13.33

As you can see, the maximum value of Z is 13.33,

Reason (R) is true: as it is the corner point theorem

6

Assertion (A): For the LPP Z= 3x+2y, subject to the constraints x+2y$\leq$2; x $\geq$0;y$\geq$ 0

both maximum value of Z and Minimum value of Z can be obtained.

Reason (R): If the feasible region is bounded then both maximum and

minimum values of Z exists.

ANSWER :a

Explanation:

- Assertion (A) is true. The feasible region is bounded and the corner points are (0,0), (2,0),(0,1)

Maximum value = 6 at (2,0) , minimum value = 0 at (0,0)

- Reason (R) is also true.. This is the corner point theorem in linear programming.

7 **Assertion (A):** For an objective function Z= 15x + 20y, corner points are

(0,0), (10,0), (0,15) and (5,5). Then optimal values are 300 and 0

respectively.

Reason (R):The maximum or minimum value of an objective function is known

as optimal value of LPP. These values are obtained at corner points.

ANSWER :a

Explanation:

- Assertion (A) is true.

Maximum value = 300 at (20,15) , minimum value = 0 at (0,0)

- Reason (R) is also true.. This is the corner point theorem in linear programming.

EXERCISE(UNSOLVED)

1 **Assertion (A):** If an LPP attains its maximum value at two corner points of

the feasible region then it attains maximum value at infinitely many points.

Reason (R): if the value of the objective function of a LPP is same at two

a corner then it is same at every point on the line joining two corner points.

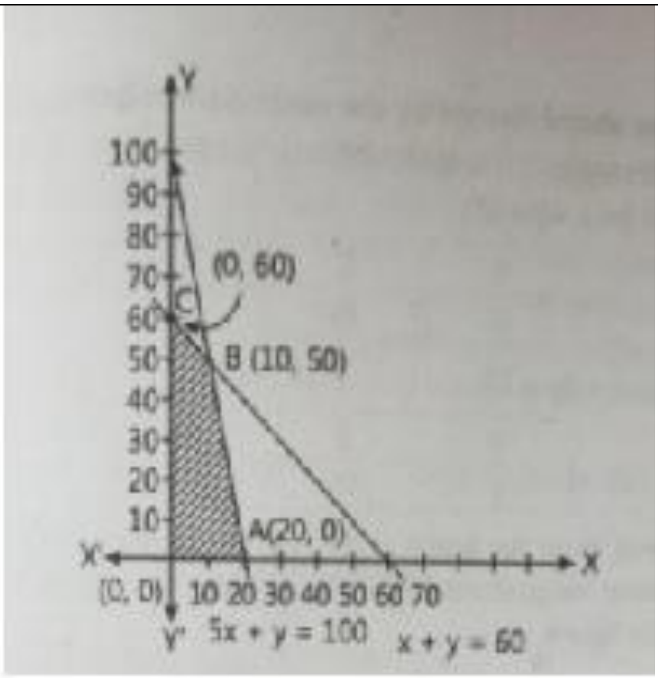
ANSWER :a

2 Consider, the graph of constraints stated as linear inequalities as below:

$5x + y \leq 100$,

$x + y \leq 60$,

$x, y \geq 0.$



Assertion (A): The points (10, 50), (0, 60) , (10,10) and (20,0) are feasible solutions.

Reason (R): Points within and on the boundary of the feasible region represent feasible solutions of the constraints.

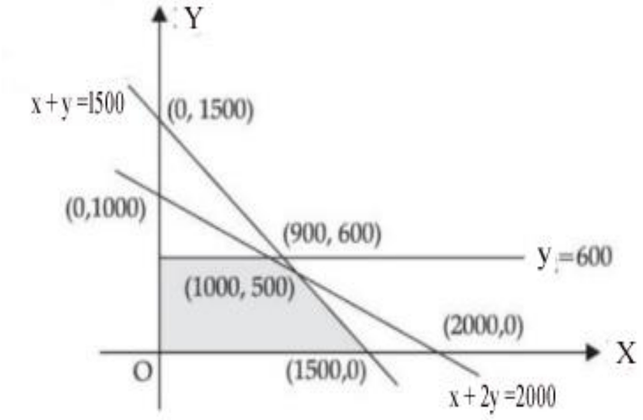
ANSWER :a

3 **Assertion (A):** For the constraints of a LPP problem given by

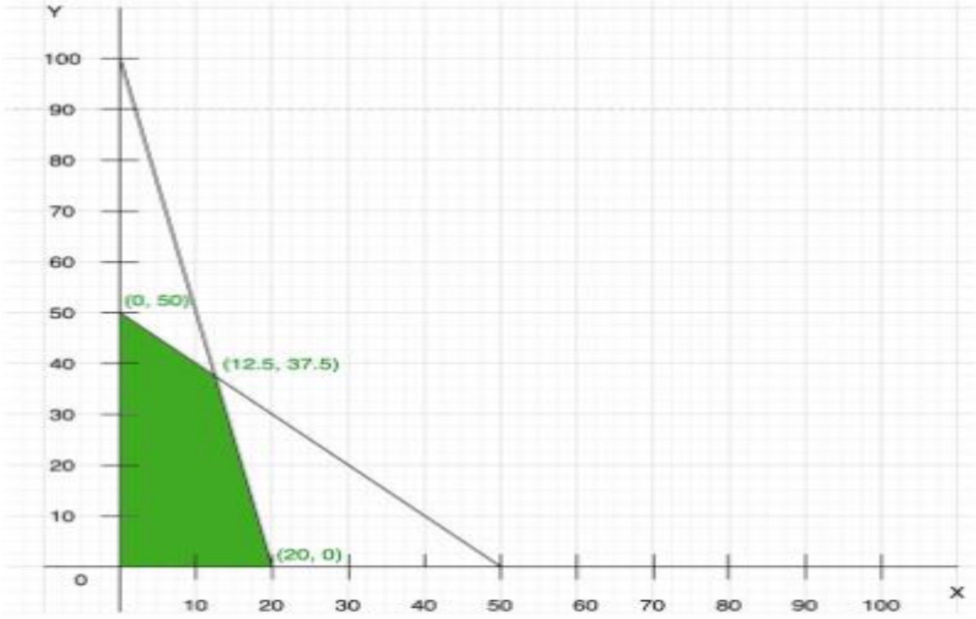
$x + 2y \leq 2000$, $x + y \leq 1500$, $y \leq 600$ and $x, y \geq 0$ the points (1000, 0), (0, 500),

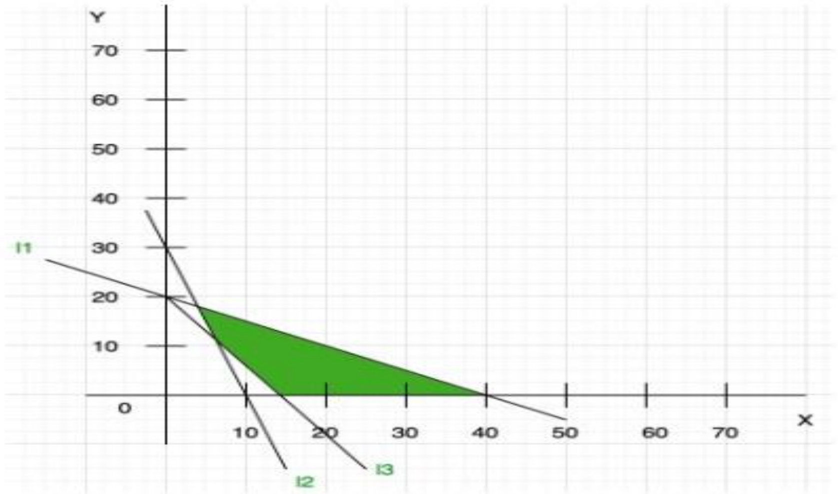
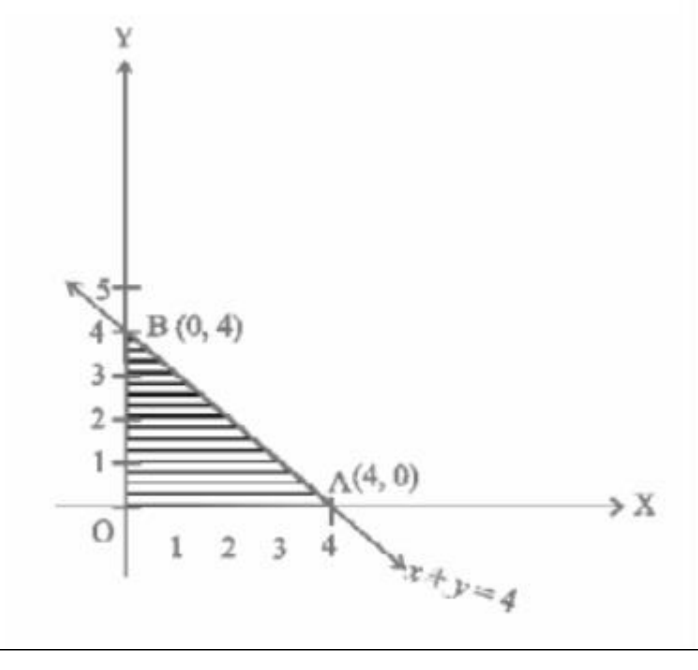
(2, 0) lie in the positive bounded region, but point (2000, 0) does not lie in the

positive bounded region.

	Reason (R):  **ANSWER :a**

VERY SHORT ANSWER QUESTIONS

1	Draw the graph of linear programming problems graphically: Maximize: Z = 50x + 15y Constraints are 5x + y ≤ 100, x + y ≤ 50, x ≥ 0, y ≥ 0 **Answer:** 

2	Minimize: $Z = 20x + 10y$ Constraints are, $x + 2y \le 40$, $3x + y \ge 30$, $4x + 3y \ge 60$, $x \ge 0$, $y \ge 0$ Draw the feasible region **Answer:** 
3	Solve the following LPP graphically: Maximise $Z = 2x + 3y$, subject to $x + y \le 4$, $x \ge 0$, $y \ge 0$ draw the graph and find the corner points **Answer:** 

4	What are the constraints and non negative restrictions? **Answer:** The inequations or equations in the variables of lpp which describe the conditions under which the optimization $(Maximization\ or\ Minimization)$ is to be accomplished are called constraints. The constraints which describe that the variables involved in the LPP are non negative
5	Maximise Z= x + 2y subject to x + y ≥5, x≥0, y≥0 **Answer:** Draw the graph. Corner points are $(0,5)$ and $(5,0)$. The values of Z at these points are 5 and 10 and max value is 10
6	Maximise Z= 4x + y subject to x + y ≤50, x≥0, y≥0 **Answer:** Draw the graph. Corner points are $(0,50)$ and $(50,0)$. The values of Z at these points are 50 and 200 and max value is 200
7	Write what is feasible region and optimal Solution **Answer:** The common region determined by all the constraints is called feasible region. Any point in the feasible region which gives the optimal value is called optimal Feasible solution
8	What are the different types of LPP and basic assumptions of LPP ? **Answer:** Manufacturing Problem , Diet Problem, Transportation Problem, Assignment Problem Certainty, Divisibility, Proportionality, Additivity
9	Explain the method of ISO profit/ ISO cost method . **Answer:** Draw the graph and identify feasible region. Give some convenient value for Z and draw the graph. after that draw a parallel line through the farthest point of

feasible region and if there is a common point for feasible region and that line, it is considered to be the point of maxima . in case of minima we take the nearest point

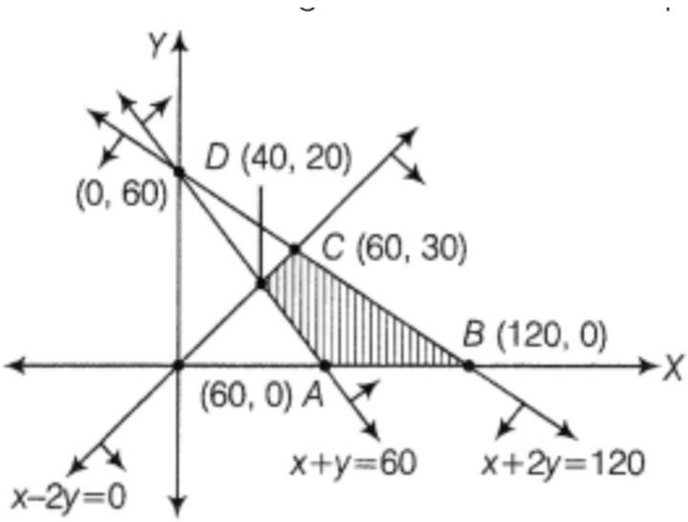
10

Draw the graph of Linear Programming Problem and show the feasible region

Maximise $Z = 5x + 10y$ subject to the constraints $x + 2y \le 120$, $x + y \ge 60$,

$x - 2y > 0$

Answer:



SHORT ANSWER QUESTIONS

Q.No. (Solved Questions)

1. 1 Solve the following linear programming problem graphically:

Maximize $Z = 4x + y$ ... (1)

subject to the constraints:

$x + y \le 50$ ... (2)

$3x + y \le 90$ ... (3)

$x \ge 0$, $y \ge 0$ ... (4)

SOLUTION :

Objective function: Z=4x+y

1. For x+ y=50 For region

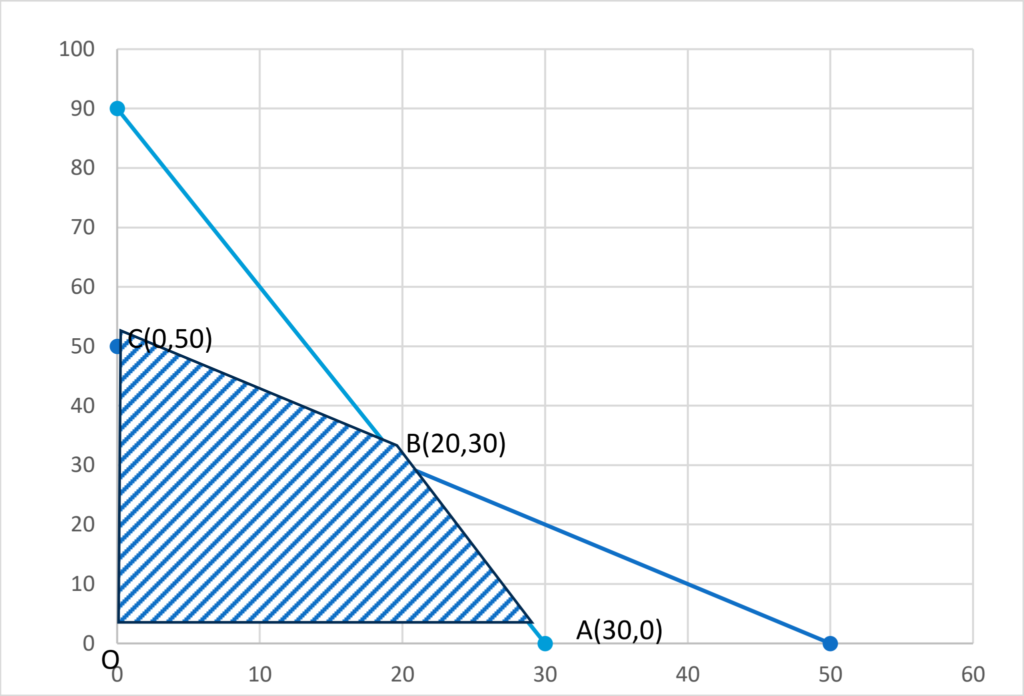
$x + y \le 50$ towards origin

x	Y
0	50
50	0

2. For 3x+ y=90 For region

$x + y \leq 50$ towards origin

x	Y
0	90
30	0



Corner point	Z=4X+Y
O(0,0)	0
A(30,0)	120
B(20,30)	110
C(0,50	50

As Feasible region bounded (so by corner point theorem)

Hence, maximum value of Z is 120 at point (30,0)

2. 2 Solve the following linear programming problem graphically:
Minimize Z = 200 x + 500 y ... (1)
subject to the constraints:
$x + 2y \geq 10$ ... (2)
$3x + 4y \leq 24$ ... (3)
$x \geq 0, y \geq 0$ ... (4)

SOLUTION :

Objective function: Z=200x+500y

1. For x+ 2y=10 For region

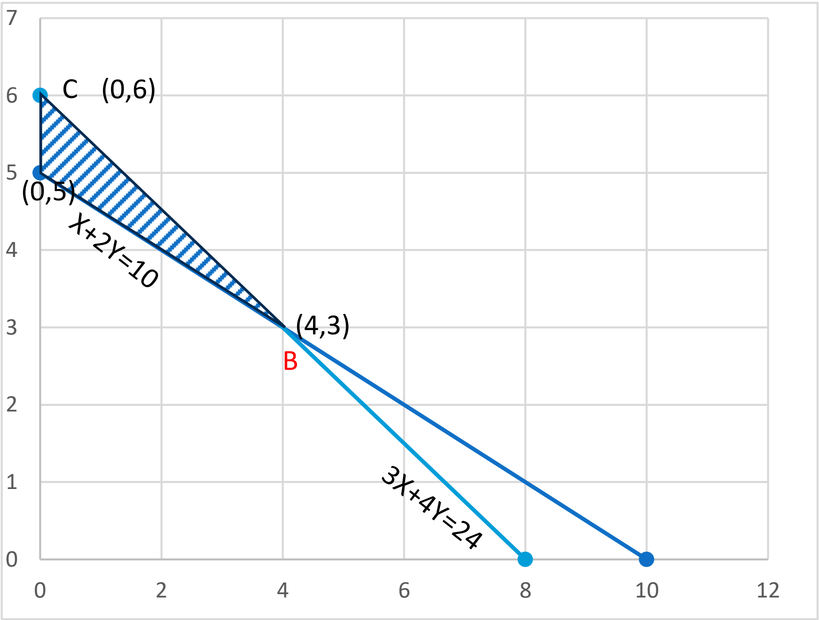
$x + 2y \geq 10$ Against origin.

x	Y
0	5
10	0

2. For 3x+ 4y=24 For region

$3x + 4y \leq 24$ towards origin

x	Y
0	6
8	0



Corner point	Z=4X+Y
A(0,5)	2500
B(4,3)	2300
C(0,6)	3000

As Feasible region bounded (so by corner point theorem)

Hence, MINIMUM value of Z is 2300 at point (4,3)

3. Solve the following problem graphically:

Maximize $Z = 3x + 9y$ ... (1)

subject to the constraints: $x + 3y \leq 60$ ... (2)

$x + y \geq 10$ ... (3) ,$x \leq y$ ... (4) , $x \geq 0, y \geq 0$ ... (5)

SOLUTION :

Objective function: Z=3x+9y

1. For x+3y=60 For region

$x + y \leq 60$ towards origin

x	y

0	20
60	0

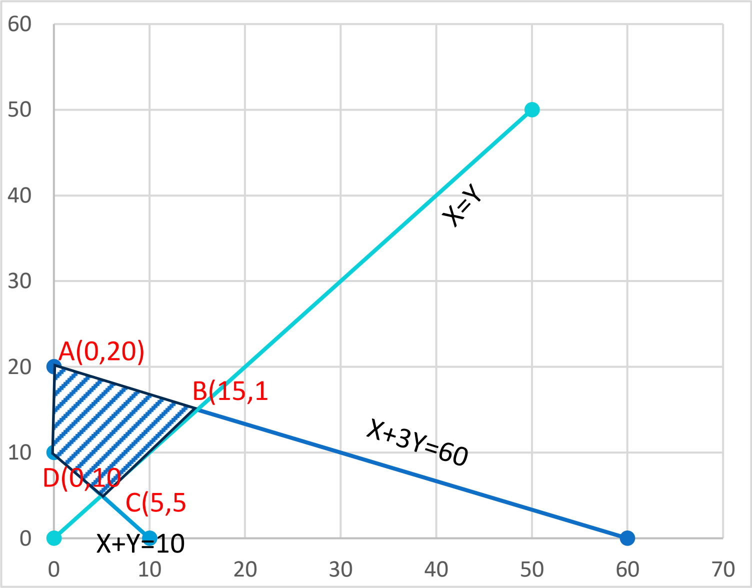
2. For x+ y=10 For region

x + y ≥ 10away origin

x	y
0	10
10	0

3.x=y checking from point (1,1)

x	y
0	0
8	8



Corner point	Z=3X+9Y
A(0,20)	180
B(15,15)	180

C(5,5)	60
D(0.10)	90

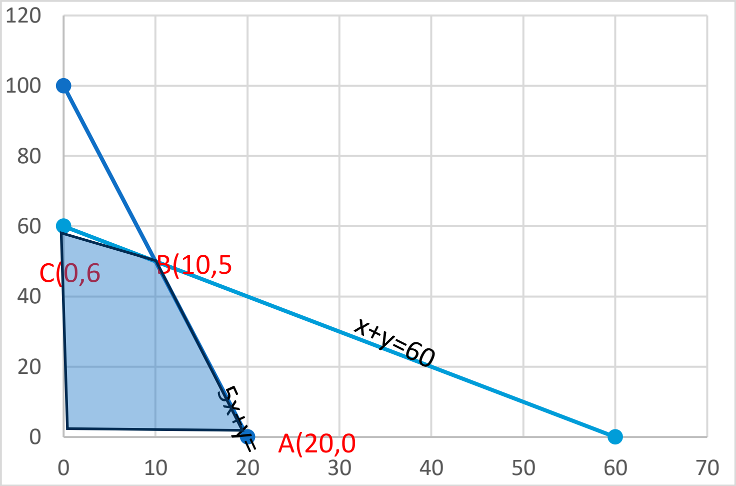
As Feasible region bounded (so by corner point theorem)

Hence, MAXIMUM value of Z is 180 at every point of line joining (15,15) to (0,20)

4. A furniture dealer deals in only two items–tables and chairs. He has Rs 50,000 to invest and has storage space of at most 60 pieces. A table costs Rs 2500 and a chair Rs 500. He estimates that from the sale of one table, he can make a profit of Rs 250 and that from the sale of one chair a profit of Rs 75. He wants to know how many tables and chairs he should buy from the available money so as to maximise his total profit, assuming that he can sell all the items which he buys.

SOLUTION :

Let x be number of table and y be chair
Objective function Z= 250x+75y
Constraints 5x+y<=100 (1)
x +y<=60 (2)
x,y>= 0 (3)



1. For 5x+ y=100 For region

5x + y ≤ 100 towards origin.

x	y
0	100
20	0

2. For x+ y=60 For region

$x + y \le 60$ towards origin

x	y
0	60
60	0

Corner point	Z=250X+75Y
A(20,0)	5000
B(10,50)	6250
C(0,60)	4500
O(0,0)	0

As Feasible region bounded (so by corner point theorem)

Hence, MAXIMUM value of Z is 6250 at (10,50)

5. Maximise $Z = 3x + 2y$ subject to $x + 2y \le 10$, $3x + y \le 15$, $x, y \ge 0$

SOLUTION :

Objective function Z=3x+2y

1. For x+ 2y=10 For region

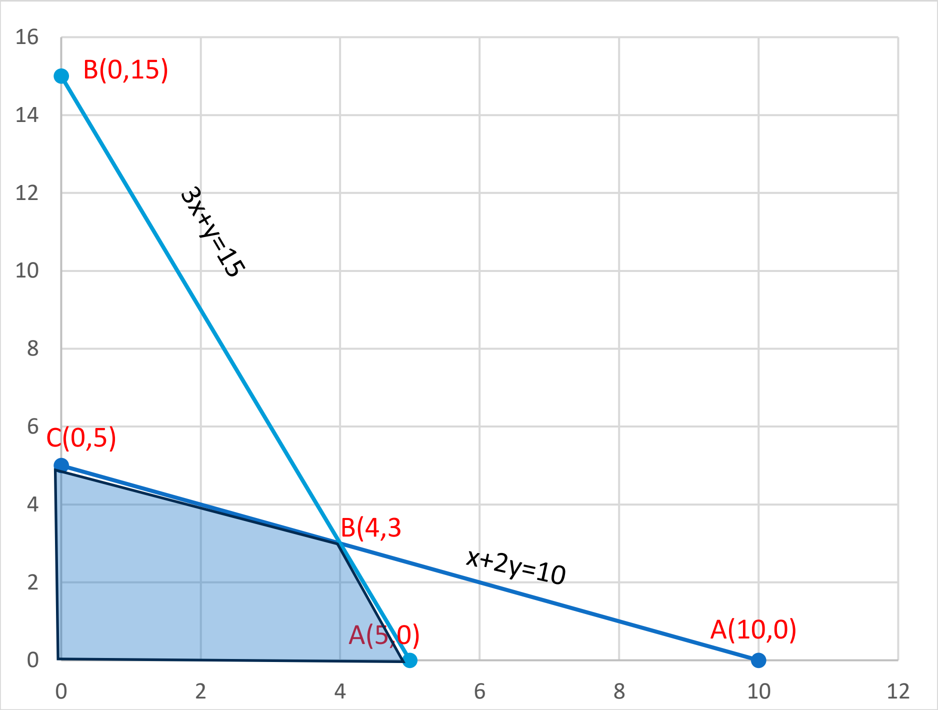
$x + 2y \le 10$ towards origin.

x	y
0	5
10	0

2. For 3x+ y=15 For region

$3x + y \le 15$ towards origin

x	y
0	15
5	0



Corner point	Z=3X+2Y
A(5,0)	15
B(4,3)	18
C(0,5)	10
O(0,0)	0

As Feasible region bounded (so by corner point theorem)

Hence, MAXIMUM value of Z is 18 at (4,3)

6. Minimize Z = x + 2y
Subject to constraints:
x >= 0 and y >= 0
2x + y >= 3
x + 2y >= 6
Show that minimum Z has more than two optimal solutions.

SOLUTION :

Minimise Z=x+2y

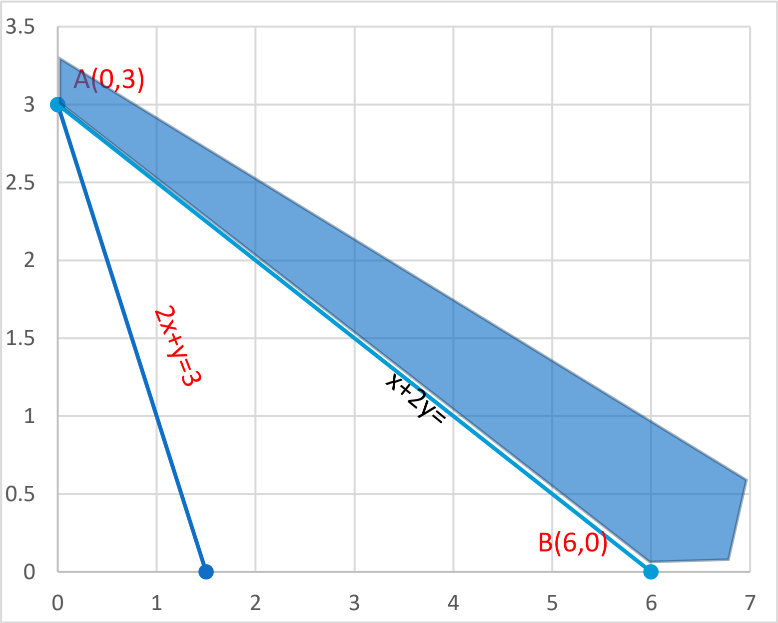
1. For 2x+ y=3 For region

2x + y ≥ 3 away to origin.

x	y
0	3
1.5	0

2. For x+ 2y=6 For region

x + 2y ≥ 6 away to origin



x	y
0	3
6	0

Corner point	Z=X+2Y

A(0,3)	6
B(6,0)	6

As Feasible region unbounded (so by corner point theorem)

Hence, minimum value may or may not be

For Z<m

X+2y<6 has no point common with feasible region.

So minimum value of Z will be 6 at every point A to B.

7.

Solve the following Linear Programming Problem Graphically.
Maximize Z = 6x + y
Subject to constraints:
$2x + y \geq 3$
$y - x \geq 0$
$x \geq 0$, and $y \geq 0$

SOLUTION:

Objective function Z=6x+y

1. For 2x+ y=3 For region

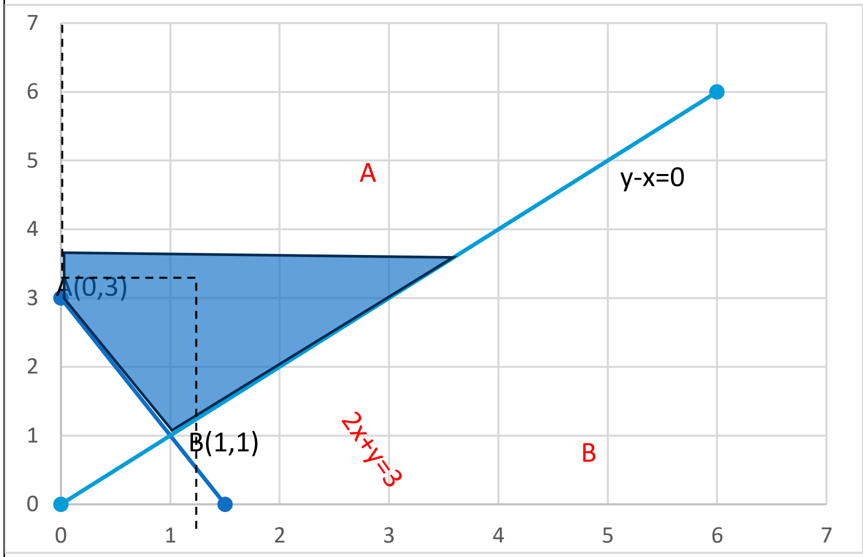
$2x + y \geq 3$ away to origin.

x	y
0	3
1.5	0

2. For x- y=0 For region

Away (1,0)

x	y
0	0
6	6



Corner point	Z=6X+2Y
A(0,3)	3
B(1,1))	7

As Feasible region unbounded (so by corner point theorem)

Hence, maximum value may or may not be

For Z>M

6X+y>7has common point with feasible region .

Hence no maximum point

8\. Maximize $Z = x + y$

Subject to constraints:

$x, y \geq 0$ $x - y \leq -1$ $x \geq y$

SOLUTION :

Objective function Z=x+y

1\. For x- y=-1 For region

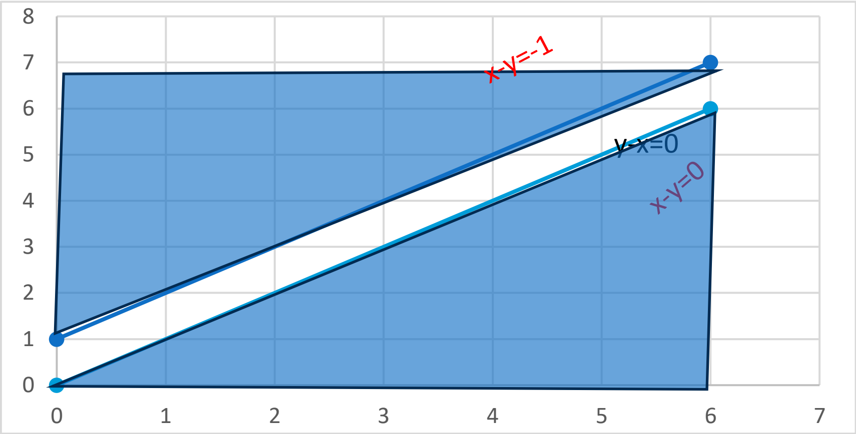
$x - y \geq -1$ away to origin.

x	y
0	1
6	7

2. For x- y=0   For region

Towards (1,0)

x	y
0	0
6	6



Plotting the graph, we can see that there is no feasible region for the constraints

Hence the given LPP has no solution and Z cannot be maximized

9. Solve the following Linear Programming Problem Graphically.
Minimize Z = 18x + 10y
Subject to   4x + y ≥20 ,2x + 3y ≥ 30
and x ≥ 0 and y ≥ 0

SOLUTION :

Objective function Z=18x+10y
1. For 4x+y=20   For region

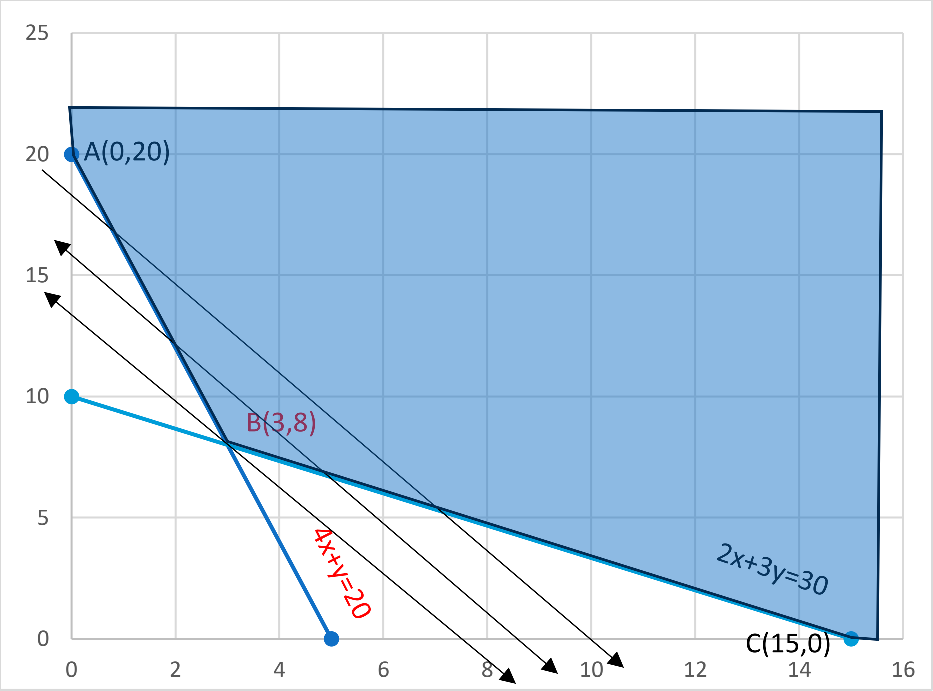
4x +y ≥ 20 away to origin.

x	y
0	20
5	0

2. For 2x+3y=30 For region

2x+3y≥ 30 away to origin

x	y
0	10
15	0



Give a value, say 180 equal to (2 times LCM of 18 and 10) to Z to obtain the line 18x + 10y =180. This line meets the co-ordinate axes at (10, 0) and (0, 18).
Putting x = 3 and y = 8 in the objective function Z = 18x + 10y, we get Z= 134 The minimum value of Z is 134 at x = 3 and y = 8 .

10. Minimize Z = 3x + 5y
Subject to constraints:
x, y ≥ 0
x + 3y – 3 ≥ 0

x + y – 2 ≥ 0

SOLUTION :

Objective function Z=3x+5y

1. For x+3y=3   For region

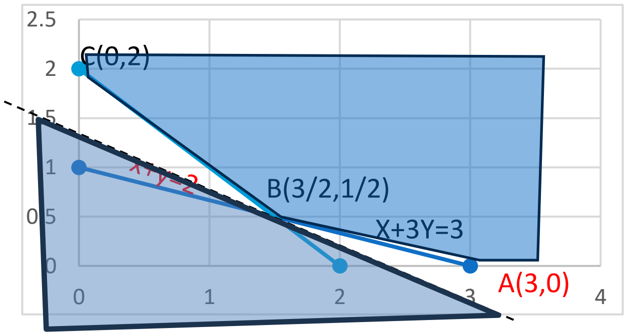
x +3y ≥ 3 away to origin.

x	y
0	1
3	0

2. For x+y=2   For region

x+y≥ 2 away to origin

x	y
0	2
2	0



Corner point	Z=3X5Y
C(0,2)	10
B(3/2,1/2))	7
A(3,0)	9

As Feasible region unbounded (so by corner point theorem)

Hence, minimum value may or may not be

For Z<m

3x+5y<7has no point common with feasible region .

Hence minimum value of Z is 7 at (3/2,1/2)

EXERCISE(UNSOLVED QUESTIONS)

1. Minimize $Z = x + 2y$ subject to $2x + y \geq 3$, $x + 2y \geq 6$, $x, y \geq 0$

2.. Minimize $Z = 5x + 10y$ subject to $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x, y \geq 0$

3. Minimize $Z = x + 2y$
subject to $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$; $x, y \geq 0$.

4. Maximize, $Z = 3x + 2y$
Subject to the constraints:
$-2x + y \leq 1$,
$x + y \leq 3$,
$x \leq 2$,
and $x \geq 0$, $y \geq 0$

5. Maximize $Z = x + y$, subject to $x - y \leq -1$, $-x + y \leq 0$, $x, y \geq 0$.

LONG ANSWER QUESTIONS

(Solved Questions)

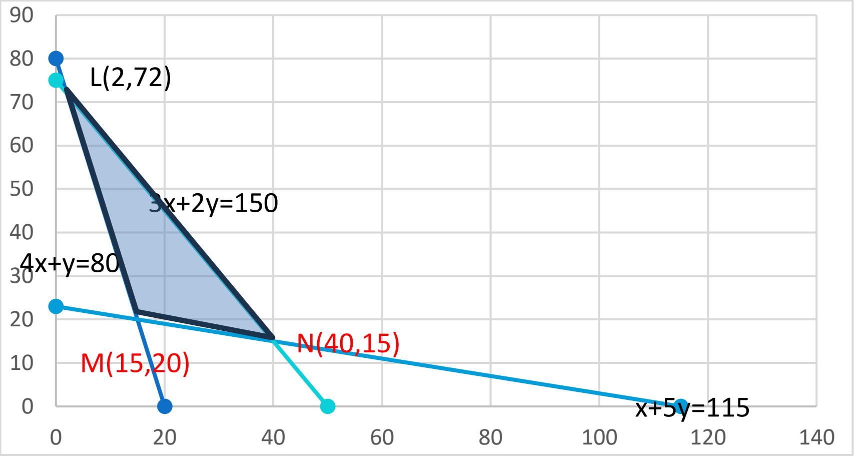
1. (Diet problem) A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires at least 240 units of calcium, at least 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

SOLUTION:

Let x and y be the number of packets of food P and Q respectively. Obviously

Minimize $Z = 6x + 3y$ (vitamin A)
subject to the constraints
$4x + y \geq 80$ ... (1)
$x + 5y \geq 115$ ... (2)
$3x + 2y \leq 150$ ... (3)

$x \geq 0, y \geq 0 \ldots (4)$



Objective function Z=6x+3y

1. For 4x+y=80 For region

$4x + y \geq 80$ away to origin.

x	y
0	80
20	0

2. For x+5y=115 For region

$x + 5y \geq 115$ away to origin

x	y
0	23
115	0

3. For 3x+2y=150 For region

$3x + 2y \leq 150$ towards origin

x	y
0	75
50	0

Corner point	Z=6X+3Y
L(2,72)	228
M(15,20))	150
N(40,15	285

As Feasible region bounded (so by corner point theorem)

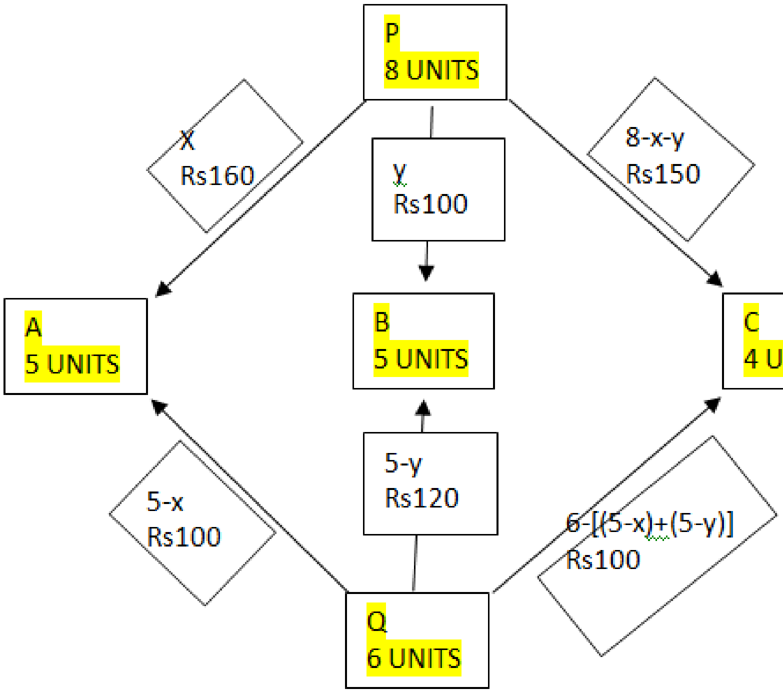
Hence, minimum value of Z will be 150 at (15,20)

2. (Transportation problem) There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:

From/TO	COST (IN Rs)		
	A	B	C
	160	100	150
	100	120	100

SOLUTION:

The problem can be explained diagrammatically as shown. Let x units and y units of the commodity be transported from the factory at P to the depots at A and B respectively. Then (8 – x – y) units will be transported to depot at C .



Total transportation cost Z is given by

$Z = 160\ x + 100\ y + 150\ (8 – x – y) + 100\ (5 – x) + 120\ (5 – y) + 100\ (x + y – 4)$

$= 10\ (x – 7\ y + 190)$

Minimize $Z = 10\ (x – 7y + 190)$

subject to the constraints:

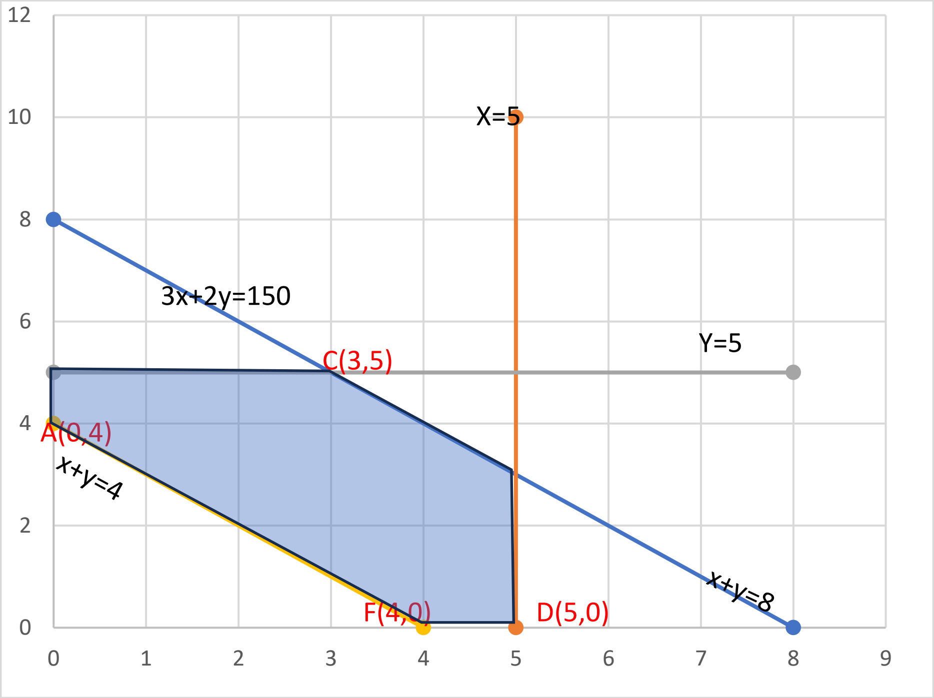
$x \geq 0, y \geq 0$ ... (1)

$x + y \leq 8$ ... (2)

$x \leq 5$ ... (3)

$y \leq 5$ ... (4)

and $x + y \geq 4$ (5)



Corner point	Z=10(x-7y+190)
(0,4)	1620
(0,5)	1550
(3,5)	1580
(5,3)	1740
(5,0)	1950
(4,0)	1940

As Feasible region bounded (so by corner point theorem)

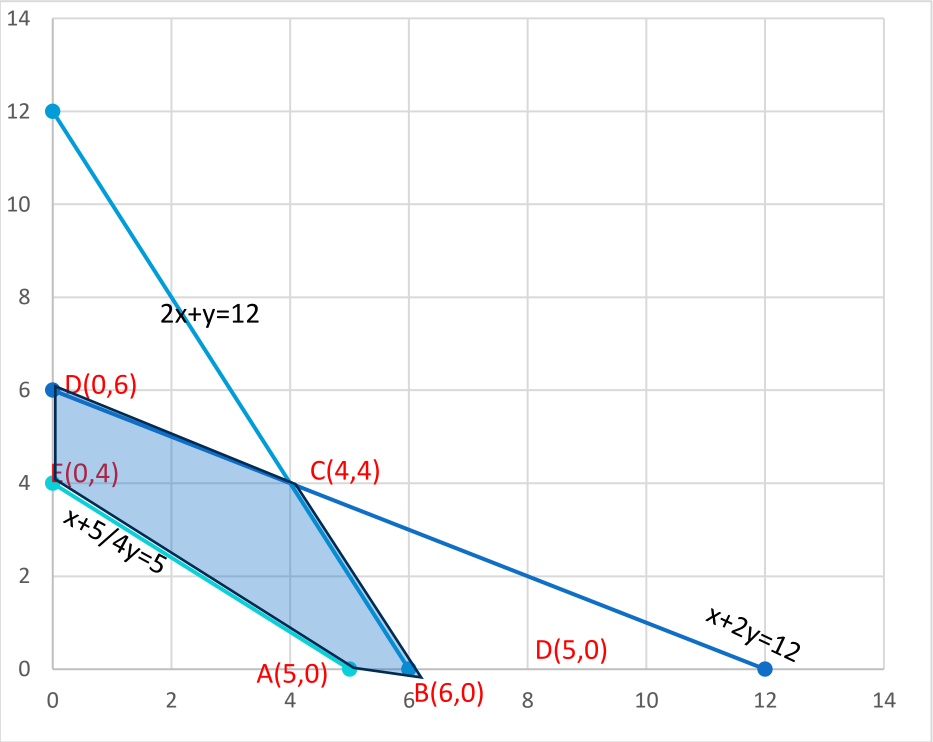
Hence, minimum value of Z is 1550 at (0,5)

3. 0 (Manufacturing problem) A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for at least 5 hours a day. She produces only two items M and N each requiring the use of all the three machines. The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table

items	Number of hours required on machines		
	I	II	III
M	1	2	1
N	2	1	1.25

SOLUTION :

Let x and y be the number of items M and N respectively.
Total profit on the production = Rs (600 x + 400 y)
Maximise Z = 600 x + 400 y
subject to the constraints:
$x + 2y \leq 12$ (constraint on Machine I) ... (1)
$2x + y \leq 12$ (constraint on Machine II) ... (2)
$x + 5/4y \geq 5$ (constraint on Machine III) ... (3)



Corner point	Z=600X+400Y
(5,0)	3000
(6,0)	3600
(4,4)	4000
(0,6)	2400
(4,0)	1600

As Feasible region bounded (so by corner point theorem)

Hence, minimum value of Z is 4000 at (4,4)

EXERCISE (UNSOLVED QUESTIONS)

1. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	Vitamin A	Vitamin B	Vitamin c
X	1	2	3
Y	2	2	1

One kg of food X costs Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet

2. A toy company manufactures two types of dolls, A and B. Market research and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of Rs 12 and Rs 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit

3.

Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

Transportation cost per quintal (in Rs)		
From/To	A	B
D	6	4
E	3	2
F	2.50	3

How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

CASE BASE QUESTIONS

(SOLVED QUESTIONS)

1.

An aeroplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. The company director want to calculate the number each class tickets to get maximum Profit using LPP. Company assumes x, y as number executive and economy class tickets respectively.

From above information solve the following questions.

(i) Find objective function. (1 mark)
(ii) Find linear inequality using “An aeroplane can carry a maximum of 200 passengers”(1 mark)
(iii) Find linear inequality using “, at least 4 times as many passengers prefer to travel by economy class than by the executive class”. (2 marks)

OR

In graphical solution , region is bounded and corner point are (20,180), (40,160) and (20,80) . Find maximum profit . (2 marks)

ANSWER:

	(i) $Z= 1000x+600y$. (ii) $x+y\leq 200$ (iii) $y\geq 4x$OR $Z= 136000$.
2.	A merchant plans to sell two types of personal computers – a desktop model and a portable model that will cost Rs 25000 and Rs 40000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Merchants wants To get maximum profit. He can not invest more than 70 lakhs.He gets profit on the desktop model is Rs 4500 and on portable model is Rs 5000. He uses graphical method to solve LPP and finds corner points of bounded region (0,175) ,(200, 50) ,(250.0) ,and (0,0)  From above information solve the following questions. (i) Find objective function . (ii) Find linear inequality “He estimates that the total monthly demand of computers will not exceed 250 units”. (iii) Find linear inequality,”Hecan not invest more than 70 lakhs “. OR Find maximum profit. **ANSWER:** (i) $Z= 4500x+5000y$ (ii) $x+y\leq 250$ (iii) $5x+8y\leq 1400$ OR RS.1150000
3.	A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs17.50 per package on nuts and Rs 7.00 per package on boltsand to maximize his profit, he operates his machines for at the most 12 hours a day? He uses graphical method to solve this LPP using x, y as nuts and bolts respectively and find that region is bounded with corner points (0,4), (3,3) ,(4,0),and (0,0) .

From above information solve the following questions.
(i) Find objective function.
(ii) Find linear inequalities“, he operates his machines for at the most 12 hours a day?
(iii) Find maximum profit.

ANSWER:
(i) $Z=17.5x+7y$
(ii) $x+3y\leq12$ and $3x+y\leq12$
(iii) maximum profit Rs. 73.50

(Unsolved Questions)

1. One kind of cake requires 200g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. One baker wants maximum number of cakes which can be made from 5kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.
He uses graphical method to solve this LPP using x, y as cakes one type and cake of other types respectively and find that feasible region bounded with corner points (0,20) ,(20,10) , (25,0) and (0,0)

From above information solve the following questions.

(i) Find objective function. (1 mark)

(ii) Find linear inequalities. (2 mars)

(iii) Find maximum number of cakes (1 mark).

2\. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftsman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftsman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time. If in this LPP one solves by graphical method and finds corner points of feasible region are as (0,0),(0,14), (4,12) and (8,0) .

From above information solve the following questions.
(i) Find objective function.
(ii) Find linear inequalities
(iii) Find maximum profit.